Homogeneous Control Systems Toolbox for MATLAB

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Abstract

Homogeneous Systems Control Toolbox (HCS Toolbox) for MATLAB is a collection of functions for design and tuning of control systems with improved control quality (faster convergences, better robustness, smaller overshoots, etc) based on the concept of a dilation symmetry (homogeneity). Homogeneous controllers/observers design well as procedures for upgrading of existing linear controllers/observers to nonlinear (homogeneous) ones are developed for both Single-Input Single-Output (SISO) and Multiply-Input Multiply-Output (MIMO) systems.

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Chapter 1

Get Started

1.1 General Context

The HCS Toolbox supports an model-based control systems design under the conventional framework presented at Fig. 1.1:

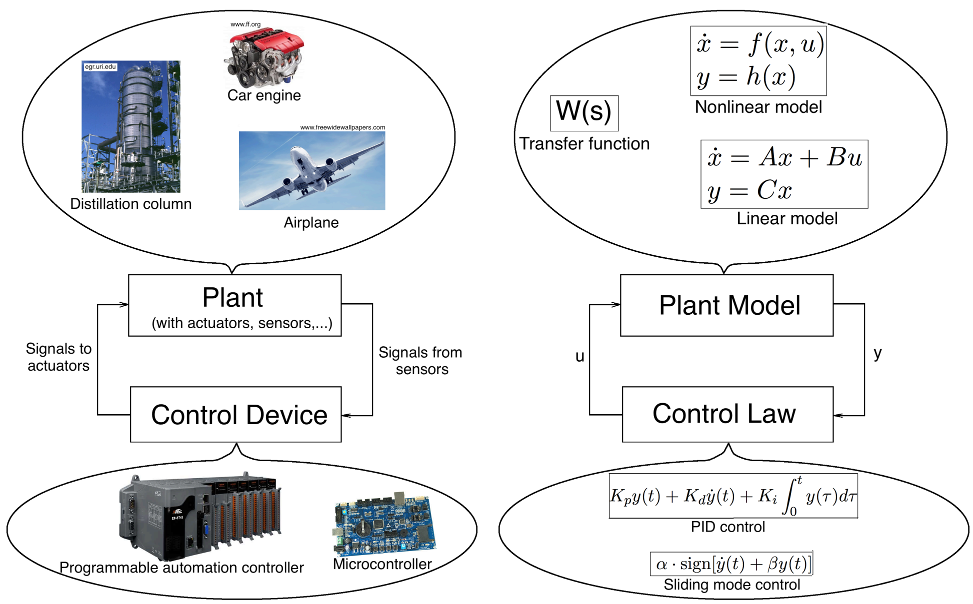

Figure 1.1: Model-based control systems design

where $x \in \mathbb{R}^n$ - the system state, $u \in \mathbb{R}^m$ - the control input , $y \in \mathbb{R}^k$ - the system output, the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{k \times n}$ are assumed to be known.

Control problems which can be solved using HCS toolbox

- design of a state feedback law u for robust finite[1]/fixed-time[2] stabilization of the state or the output (i.e., $y(t) = \mathbf{0}, \forall t \geq T$)
- upgrading a linear stabilizing controller (e.g., $u = K_{lin}y$) to a nonlinear one with a better quality
- design of a dynamic observer[3] for finite/fixed-time estimation of the state x based on the measured output y
- upgrading a linear (Lunberger) observer of x to a nonlinear one with a better quality
- design of an observer-based feedback law for robust finite/fixed-time stabilization of the system state to zero using the output measurements only

The HCS toolbox solves the above design problems based on the concept of the **generalized homogeneity** (see Chapter 2) and contains (see Chapter 3)

- `m-functions` for homogeneous controllers/observers design
- `m-functions` for discretization of controllers/observers [4]
- examples of homogeneous systems design realized in `m-files`
- demo of upgrading linear controller/observer for the rotary inverted pendulum[5] to homogeneous one with a better quality.
- various `m-functions` for homogeneous systems

[1]Finite-time stability $\Rightarrow \forall x(0) \in \mathbb{R}^n, \exists T = T(x(0)) \geq 0 : x(t) = \mathbf{0}, \forall t \geq T$

[2]Fixed-time stability $\Rightarrow \exists T_{\max} > 0 : x(t) = \mathbf{0}, \forall t \geq T_{\max}, \forall x(0) \in \mathbb{R}^n$

[3]A system $\dot{z} = f(z, y), y = Cx$ is a finite-time observer of x if $z(t) = x(t)$ for $t \geq T$, where $T = T(x(0)) < +\infty$. The above system is a fixed-time observer if $\sup_{x(0)} T(x(0)) < +\infty$.

[4]The discretization algorithms are not optimized for digital implementation in low performance devices (e.g., outdated controllers) or for fast control plants (like power converters). Please do contact the toolbox developer `andrey.polyakov@inria.fr` if a specific algorithm of digital implementation for your concrete industrial application is required

[5]The demo is supported by real physical experiments with Rotary Inverted Pendulum Quanser QUBE Servo-2 connected to ControlHub platform (http://valse-pendulum.lille.inria.fr:5000/. The users can repeat remotely all control experiments and test/compare their own controllers/observers.

1.2 Supporting Information

Installation of the HCS Toolbox:

1 Download zip-file from http://gitlab.inria.fr/polyakov/hcs-toolbox-for-matlab

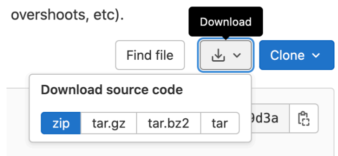

Figure 1.2: Click a button as shown above to download zip-file

2 Extract the archive

3 Add[1] path to the folder '`...\HCS_toolbox_ver01\`' in the search path of MATLAB

Type `help HCS_toolbox` in the Command Line of MATLAB to check if the installation is well done (you should see the list of functions of HCS Toolbox).

Compatibility:

- The HCS Toolbox uses just common MATLAB functions. It is compatible with most versions of MATLAB.
- The recommended browser for for ControlHub Demo

  http://valse-pendulum.lille.inria.fr:5000/

  is Google Chrome.

[1]Use 'Set Path' in the panel of MATLAB or the function `addpath('<path_to_folder>')` in the Command Line of MATLAB

Chapter 2

Mathematical Backgrounds

Notation

- $\mathbb{R}$ - field of real, $\mathbb{R}_+ = \{\alpha \in \mathbb{R} : \alpha > 0\}$;
- $|x| = \sqrt{x^\top x}$ is the Euclidean norm in $\mathbb{R}^n$, $x \in \mathbb{R}^n$
- $P \succ 0$ means $P = P^\top \in \mathbb{R}^{n\times n}$ is a positive definite matrix $(\prec 0, \succeq 0, \preceq 0)$
- $\mathbf{d}(s) = e^{aG_\mathbf{d}}, s \in \mathbb{R}$ is a linear dilation in $\mathbb{R}^n$ with an anti-Hurwitz matrix $G_\mathbf{d}$;
- $\|x\| = \sqrt{x^\top P x}$ is the weighted Euclidean norm such that $P \succ 0, PG_\mathbf{d} + G_\mathbf{d}^\top P \succ 0$;
- $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$ is the unit sphere;
- $\|\cdot\|_\mathbf{d}$ is the canonical homogeneous induced by $\|\cdot\|$(see below);
- $\mathcal{H}_\mathbf{d}(\mathbb{R}^n)$ - a set of all **d**-homogeneous functions $\mathbb{R}^n \mapsto \mathbb{R}$ (see below);
- $\mathcal{F}_\mathbf{d}(\mathbb{R}^n)$ be a set of all **d**-homogeneous vector fields $\mathbb{R}^n \mapsto \mathbb{R}^n$;
- $\deg_\mathbf{d}(f)$ - homogeneity degree of f.

2.1 Problems Under Consideration

The classical problems of the control systems theory

- state/output stabilization of a system
- state estimation of a system

are considered under additional constraint of a finite or a fixed-time response (see below) of the system. Solutions to these problems are going to be developed based on the generalized homogeneity surveyed in this chapter. Let us first theoretical formulations of the mentioned control/estimation problems.

2.1.1 Stabilization Problem

Model of control system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t > 0 \tag{2.1}$$

- $x(t) \in \mathbb{R}^n$ is the system state
- $u(t) \in \mathbb{R}^m$ is the control input (can be modified as need)
- $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are known matrices

Asymptotic Stabilization by a Feedback Law

The problem is to design $\tilde{u} : \mathbb{R}^n \mapsto \mathbb{R}^m$ such that the system

$$\dot{x}(t) = Ax(t) + B\tilde{u}(x(t))$$

is asymptotically stable[1] $\Rightarrow$

$$x(t) \to 0 \quad \text{as} \quad t \to +\infty$$

Finite-time Stabilization by a Feedback Law

The problem is to design $\tilde{u} : \mathbb{R}^n \mapsto \mathbb{R}^m$ such that the system

$$\dot{x}(t) = Ax(t) + B\tilde{u}(x(t))$$

is finite-time stable $\Rightarrow$

$$\forall x(0) \in \mathbb{R}^n, \quad \exists T = T(x(0)) \geq 0 \quad : \quad x(t) = \mathbf{0}, \quad \forall t \geq T$$

Fixed-time Stabilization by a Feedback Law

The problem is to design $\tilde{u} : \mathbb{R}^n \mapsto \mathbb{R}^m$ such that the system

$$\dot{x}(t) = Ax(t) + B\tilde{u}(x(t))$$

is fixed-time stable $\Rightarrow$

$$\exists T_{\max} > 0 \quad : \quad x(t) = \mathbf{0}, \quad \forall t \geq T_{\max}, \quad \forall x(0) \in \mathbb{R}^n$$

A stabilization of an output $y = Cx$ is a particular case of the above problems.

[1] The classical solution is $\tilde{u} = Kx$ such that $A + BK$ is Hurwitz (i.e., all its eigenvalues have negative real parts).

2.1.2 State Estimation Problem

Model of the system:

$$\dot{x}(t) = Ax(t) + p(t), \quad y(t) = Cx(t) \tag{2.2}$$

- $x(t) \in \mathbb{R}^n$ is the system state (unknown)
- $p(t) \in \mathbb{R}^n$ is the exogenous input (can be measured on-line)
- $y(t) \in \mathbb{R}^k$ is the measured output
- $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{k\times n}$ are known matrices

Asymptotic Observer

The problem is to design a system (asymptotic observer)[2]

$$\dot{z}(t) = f(z(t), p(t), y(t)), \qquad f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \mapsto \mathbb{R}^n$$

such that $\quad \|z(t) - x(t)\| \to 0 \quad$ as $\quad t \to +\infty$

Finite-time Observer

The problem is to design a system (finite-time observer)

$$\dot{z}(t) = f(z(t), p(t), y(t)), \qquad f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \mapsto \mathbb{R}^n$$

such that $\quad \forall x(0) \in \mathbb{R}^n, \quad \exists T = T(x(0)) \geq 0 \quad : \quad \|z(t) - x(t)\| = 0, \quad \forall t \geq T$

Fixed-time Observer

The problem is to design a system (fixed-time observer)

$$\dot{z}(t) = f(z(t), p(t), y(t)), \qquad f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \mapsto \mathbb{R}^n$$

such that $\quad \exists T_{\max} > 0 \quad : \quad \|z(t) - x(t)\| = 0, \quad \forall t \geq T_{\max}, \quad \forall x(0) \in \mathbb{R}^n.$

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[2] The classical solution (Luenberger observer) is $\dot{z}(t) = Az(t) + B\tilde{u}(t) + L(y(t) - Cz(t))$ such that the matrix $A + LC$ is Hurwitz (i.e., all its eigenvalues are placed in the left complex half-plane).

2.2 From Linearity to Homogeneity in Control Systems Design

The basic idea of the homogeneity based control/observer design is an expansion of linear methods/algorithms of (asymptotic controllers/observers) design to nonlinear systems such that finite/fixed-time stability can be provided by means of the tuning of the so-called homogeneity degree. This section explains the basis intuitions behind the mentioned idea.

2.2.1 Homogeneity is a Dilation Symmetry

In mathematics, an invariance of some characteristics of an object with respect to a certain group of transformations is known as a symmetry. The simplest example of a symmetry is the invariance of geometric figures with respect to a rotation or a dilation (see Fig.2.1).

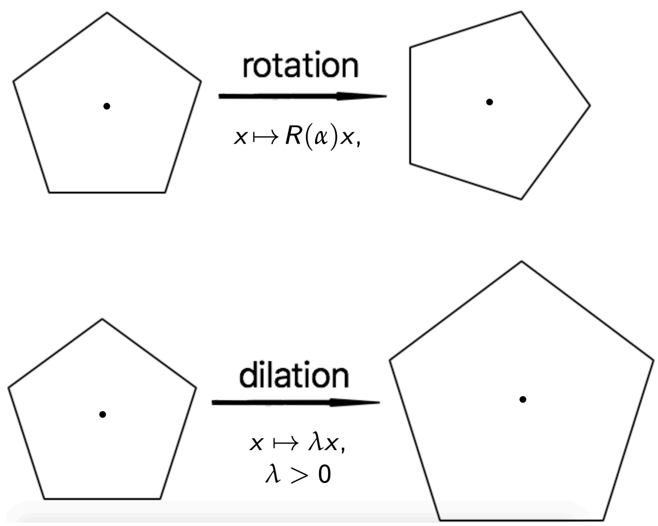


Figure 2.1: Rotation and dilation symmetries of the figure

By definition, homogeneity is a dilation symmetry. For functions, it can be identified analyzing the linearity property.

Linearity = **Homogeneity** + **Additivity** + **Central Symmetry**

f is linear $\Leftrightarrow$ $f(\lambda x) = \lambda f(x)$ & $f(x+y) = f(x) + f(y)$ & $f(-x) = -f(x)$

Example: $f(x) = x_1 + x_2$, where $x = (x_1, x_2)^\top$

Standard Homogeneity (*Leonhard Euler, 18th century*)

$x \mapsto \lambda x$ (standard dilation) $\qquad f(\lambda x) = \lambda^{\nu} f(x), \ \forall s, x$ (symmetry)

$\lambda > 0$ - scaling factor $\qquad \nu \in \mathbb{R}$ - homogeneity degree

Example: $f(x) = x_1 x_2 + x_2^2$ is standard homogeneous of degree 2:

$$f(\lambda x) = \lambda^2 f(x)$$

Generalized Homogeneity (*Zubov 1958, Kawski 1991* [24, 6])

$x \to \mathbf{d}(s)x$ (generalized dilation) $\qquad f(\mathbf{d}(s)x) = e^{\nu s} f(x)$, (symmetry)

Limit property: $\lim\limits_{s\to-\infty} \|\mathbf{d}(s)x\| = 0, \quad \lim\limits_{s\to+\infty} \|\mathbf{d}(s)x\| = +\infty, \quad \forall x \neq \mathbf{0}$

Example: $\mathbf{d}(s) = \begin{pmatrix} e^{2s} & 0 \\ 0 & e^{s} \end{pmatrix}$, $f(x) = x_1 + x_2^2$ is $\mathbf{d}$-homogeneous:

$$\mathbf{d}(s)x = \left(e^{2s}x_1, \ e^{s}x_2\right)^{\top} \quad \text{and} \quad f(\mathbf{d}(s)x) = e^{2s} f(x)$$

The HPC toolbox deals only with linear dilations in $\mathbb{R}^n$.

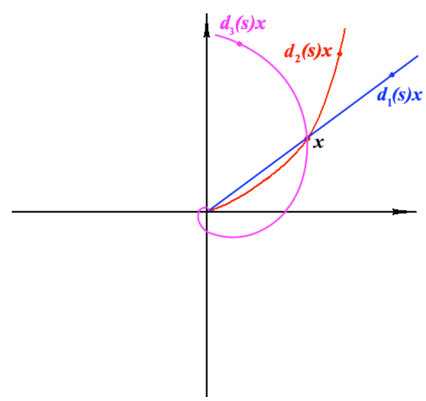
Linear Dilation (*Polyakov 2019* [14, 15])

A continuous linear **dilation** in $\mathbb{R}^n$ is a matrix-valued function given by

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}} = \sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}, \qquad s \in \mathbb{R}$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ is an *anti-Hurwitz matrix*[3] called a **generator** of $\mathbf{d}$.

- **Standard dilation**:
  $\mathbf{d}_1(s) = e^s I, \quad G_{\mathbf{d}} = I \in \mathbb{R}^{n\times n}$
- **Weighted dilation**:
  $\mathbf{d}_2(s) = \mathrm{diag}\{e^{r_i s}\}, \ G_{\mathbf{d}} = \mathrm{diag}\{r_i\} \succ 0$
- **Linear dilation**:
  $\mathbf{d}_3(s) = e^{sG_d}$, $G_{\mathbf{d}}$ is anti-Hurwitz



[3] A matrix is anti-Hurwitz if its eigenvalues have positive real parts

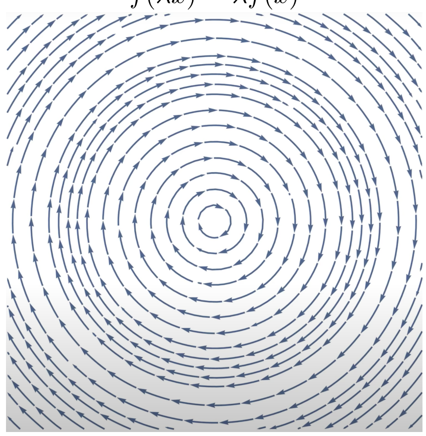
2.2.2 Properties of Homogeneous Systems

Homogeneity (dilation symmetry) of a function is inherited by any other mathematical object induced by this function: the derivatives of the homogenous functions are homogeneous as well, solutions of differential equations with homogeneous right-hand side are symmetric, etc. Indeed, for example, see below the dilation symmetry of solutions of two standard homogeneous systems (harmonic and relays oscillators) with respect to scaling of initial conditions.

Harmonic oscillator

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1, \end{cases} \quad x_1, x_2 \in \mathbb{R}$$

$$f(\lambda x) = \lambda f(x)$$



Relay oscillator

$$\begin{cases} \dot{x}_1 = -\text{sign}[x_2], \\ \dot{x}_2 = \text{sign}[x_1], \end{cases} \quad x_1, x_2 \in \mathbb{R}$$

$$f(\lambda x) = \lambda^0 f(x)$$

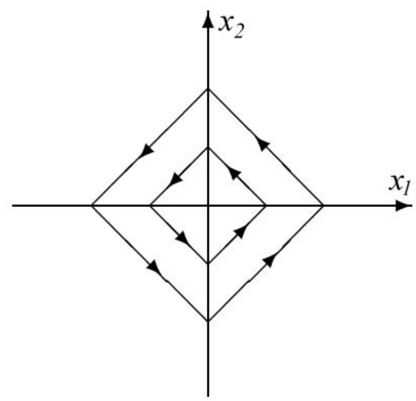


Table 1: Properties of linear vs homogeneous systems

	Linear System $\dot{x} = Ax, \ \ x(0) = x_0$ $A \in \mathbb{R}^{n\times n}$ is a matrix	**Homogeneous System** $\dot{x} = f(x), \ \ x(0) = x_0$ $f(\mathbf{d}(s)x) = e^{\mu s}\mathbf{d}(s)f(x)$
Trajectory Scaling	$x(t, e^s x_0) = e^s x(t, x_0)$	$x(t, \mathbf{d}(s)x_0) = \mathbf{d}(s)x(e^{\mu s}t, x_0)$
Local ⇔ Global	✓	✓
Invariance[4] ⇔ Stability	✓	✓
Stability ⇒ Robustness (Input-to-State Stability)	$\dot{x} = Ax + Dw$ $w \in L^\infty$	$\dot{x} = f(x, w)$ $w \in L^\infty$
Convergence Rate	Exponential	+Finite/Fixed-time ($\mu \neq 0$)
Lyapunov Function	A weighted Euclidean norm $V = \sqrt{x^\top P x}, \quad P \succ 0$	A homogeneous norm $V(\mathbf{d}(s)x) = e^s V(x)$
Consistent discretization (preserves convergence rate)	✓ Exponential	✓ +Finite/Fixed-time ($\mu \neq 0$)

Question

Is there any potential advantage of a homogeneous system versus a linear one?

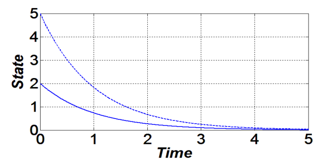
[4] ∃ a positively invariant compact set

2.2.3 Linearity vs Homogeneity

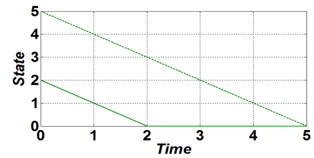
I. **Convergence rates of homogeneous and locally homogeneous systems:**

$$\begin{cases} \dot{x}(t) = u(t), \\ x(0) = x_0, \end{cases} \qquad x, u \in \mathbb{R}.$$

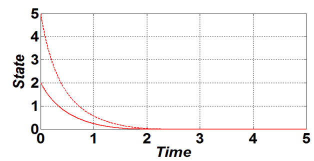
Exponential stability (*Lyapunov 1892*, [8]):
$u(t) = -x(t)$
$x(t) = e^{-t}x_0 \to 0$ if $t \to +\infty$



Finite-time stability (*Roxin 1966*, [21]):
$u(t) = -\mathrm{sign}(x(t))$
$x(t) = 0$ for $t \geq \|x_0\|$



Fixed-time stability (*Polyakov 2012*, [13]):
$u(t) = -\left(|x(t)|^{\frac{1}{2}} + |x(t)|^{\frac{3}{2}}\right)\mathrm{sign}(x(t))$
$x(t) = 0$ for $t \geq \pi$ independently of x_0



Conclusion I

Homogeneous system may have faster convergence than linear one.

II. **Robustness issues of homogeneous systems**:

Model of system	$\dot{x} = \lambda x + u$, where $\lambda > 0$ is unknown constant	$\dot{x} = u + g(t)$, where $g(t)$ is unknown but bounded function $\lvert g(t)\rvert \leq \bar{g}$.
Control aim	stabilize x to 0	stabilize x to 0
Linear control	$u = -kx$ *cannot guarantee a boundedness of solutions*	$u = -kx$ *cannot guarantee asymptotic stability*
Homogeneous control	$u = -kx^3$, $k > 0$ guarantees a **practical (fixed-time) stability** $\limsup_{t \to +\infty} \lvert x(t)\rvert = \sqrt{\lambda/k}$	$u = -(\bar{g} + k)\mathrm{sign}(x)$ guarantees **local (aympt.) stability** $x(t) = 0, \quad \forall t \geq \lvert x_0\rvert/k$

Conclusion II

Homogeneous system may be more robust than linear one.

III. **Elimination of "unbounded" peaking effect**:

Model of system:

$$\begin{cases} \dot{x} = Ax + bu(x), \\ \|x(0)\| \leq 1, \end{cases} \qquad t > 0, \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}$$

where $x = (x_1, x_2, ..., x_n)^\top$, $u : \mathbb{R}^n \mapsto \mathbb{R}$.

Control aim: $\|x(t)\| \leq \varepsilon, \quad \forall t \geq T,$ where $\varepsilon > 0$, $T > 0$ are given

- **Linear control**: For any $\varepsilon > 0$ and $T > 0$ there exists $k = (k_1, k_2, ..., k_n)$:

  $$u_\ell(x) := kx \quad \Rightarrow \quad \|x(t)\| \leq Ce^{-\sigma t} \leq \varepsilon, \quad \forall t \geq T$$

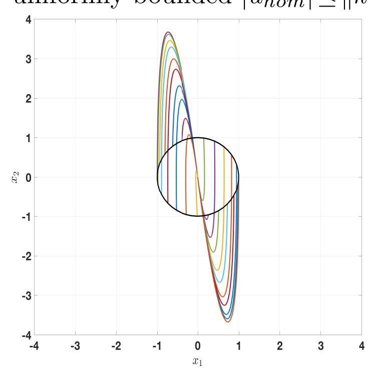
  Unbounded "peaking": There exists[5] $\gamma > 0$ independent of σ such that

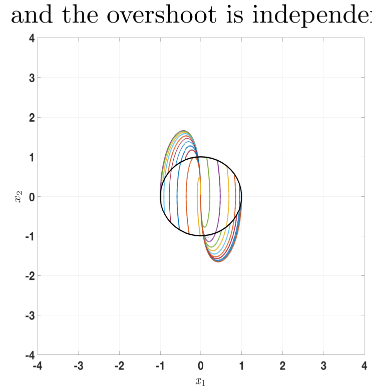
  $$\sup_{0 \leq t \leq \sigma^{-1}} \sup_{\|x(0)\|=1} \|x(t)\| \geq \gamma \sigma^{n-1} \to +\infty \text{ as } \varepsilon \to 0$$

- **Homogeneous control**: For any $T > 0$ there exists $\tilde{k} = (\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n)$:

  $$u_{hom}(x) := \tilde{k}\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x \quad \Rightarrow \quad \|x(t)\| = 0, \quad \forall t \geq T.$$

  where $\|\cdot\|_{\mathbf{d}}$ is homogeneous norm (see below). The control u_{hom} is globally uniformly bounded $|u_{hom}| \leq \|\tilde{k}\|$ and the overshoot is independent of $\varepsilon > 0$.





"Overshoots" of linear (left) and homogeneous (right) controllers ($n=2$, $\varepsilon = 0.005$, $T=1$)

Conclusion II

Homogeneous system may have smaller overshoot than linear one.

[5] Izmailov 1987 [5], Polyak & Smirnov 2016 [12]

2.3 Generalized Homogeneity

2.3.1 Linear Dilations

A **linear continuous dilation** in $\mathbb{R}^n$ is a matrix-valued function given by

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}} = \sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}, \qquad s \in \mathbb{R}$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ is an *anti-Hurwitz matrix*[6] called a **generator** of $\mathbf{d}$.

Monotone dilation

Let $\|\cdot\|$ be a norm in $\mathbb{R}^n$ and $\mathbf{d}$ be a dilation[7] in $\mathbb{R}^n$. A dilation $\mathbf{d}$ is said to be strictly monotone with respect to $\|\cdot\|$ if $\exists \beta > 0 : \|\mathbf{d}(s)\| \leq e^{\beta s}$.

The standard dilation $\mathbf{d}(s) = e^s I_n$ is always monotone due to standard homogeneity (by definition) of any norm $\|e^s x\| = e^s\|x\|$.

Criterion of monotonicity of linear continuous dilation in $\mathbb{R}^n$

A linear continuous dilation $\mathbf{d}$ is strictly monotone with respect to the weighted Euclidean norm

$$\|x\| = \sqrt{x^\top P x}, \quad x \in \mathbb{R}^n,$$

where $0 \prec P = P^\top \in \mathbb{R}^{n\times n}$, **if and only if** the following linear matrix inequality holds

$$PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, \tag{2.3}$$

where $G_{\mathbf{d}} \in \mathbb{R}^n$ is the generator of the dilation $\mathbf{d}$. Moreover, one has

$$\begin{aligned} \beta\|\mathbf{d}(s)z\|^2 \leq \frac{\frac{d}{ds}\|\mathbf{d}(s)z\|^2}{2} &\leq \alpha\|\mathbf{d}(s)z\|^2, \quad \forall s \in \mathbb{R}, \forall z \in \mathbb{R}^n, \\ e^{\alpha s} \leq \lfloor \mathbf{d}(s) \rfloor \leq \|\mathbf{d}(s)\| &\leq e^{\beta s}, \quad \forall s \leq 0, \\ e^{\beta s} \leq \lfloor \mathbf{d}(s) \rfloor \leq \|\mathbf{d}(s)\| &\leq e^{\alpha s}, \quad \forall s \geq 0, \end{aligned} \tag{2.4}$$

where

$$\alpha = \frac{1}{2}\lambda_{\max}\left(P^{\frac{1}{2}} G_{\mathbf{d}} P^{-\frac{1}{2}} + P^{-\frac{1}{2}} G_{\mathbf{d}}^\top P^{\frac{1}{2}}\right) > 0,$$

$$\beta = \frac{1}{2}\lambda_{\min}\left(P^{\frac{1}{2}} G_{\mathbf{d}} P^{-\frac{1}{2}} + P^{-\frac{1}{2}} G_{\mathbf{d}}^\top P^{\frac{1}{2}}\right) > 0.$$

[7] A dilation is a one-parameter group of transformations $\mathbf{d}(s) : \mathbb{R}^n \mapsto \mathbb{R}^n, s \in \mathbb{R}$ satisfying the limit property $\lim_{s\to-\infty} \|\mathbf{d}(s)x\| = 0, \quad \lim_{s\to+\infty} \|\mathbf{d}(s)x\| = +\infty, \quad \forall x \neq \mathbf{0}$

2.3.2 Canonical Homogeneous Norm

Vector space

A vector space over the filed $\mathbb{R}$ is a set $\mathbb{V}$ together with two operations:

- *vector addition* $\mathbb{V}\times\mathbb{V}\mapsto\mathbb{V}$ denoted by $v+w$ for $v,w\in\mathbb{V}$.
- *multiplication by a scalar* $\mathbb{R}\times\mathbb{V}\mapsto\mathbb{V}$ denoted by $\alpha\cdot v$ for $\alpha\in\mathbb{R}$, $v\in\mathbb{V}$.

satisfying certain axioms

- *Associativity*: $u+(v+w)=(u+v)+w$
- *Distributivity*: $(\alpha+\beta)u=\alpha\cdot u+\beta\cdot v$
- ...

For $\mathbb{V}=\mathbb{R}^n$ the multiplication of $v=(v_1,...,v_n)^\top\in\mathbb{R}^n$ by $\alpha\in\mathbb{R}$ is traditionally defined as follows

$$\alpha v=(\alpha v_1,...,\alpha v_2)^\top \qquad \text{(standard dilation!!!)}$$

Question

Is it possible to construct a vector space using a generalized dilation as a multiplication by a scalar?

Definition (a norm):
$\|\cdot\|\in C(\mathbb{R}^n,\mathbb{R}_+)$ is a norm if

1) $\|x\|=0\Leftrightarrow x=\mathbf{0}$

2) $\|\pm e^s x\|=e^s\|x\|$

3) $\|x+y\|\le\|x\|+\|y\|$

Definition (a homogeneous norm):
$\|\cdot\|_{\mathbf{d}}\in C(\mathbb{R}^n,\mathbb{R}_+)$ is a $\mathbf{d}$-homogeneous norm if

1) $\|x\|_{\mathbf{d}}=0\Leftrightarrow x=\mathbf{0}$

2) $\|\pm\mathbf{d}(s)x\|_{\mathbf{d}}=e^s\|x\|$

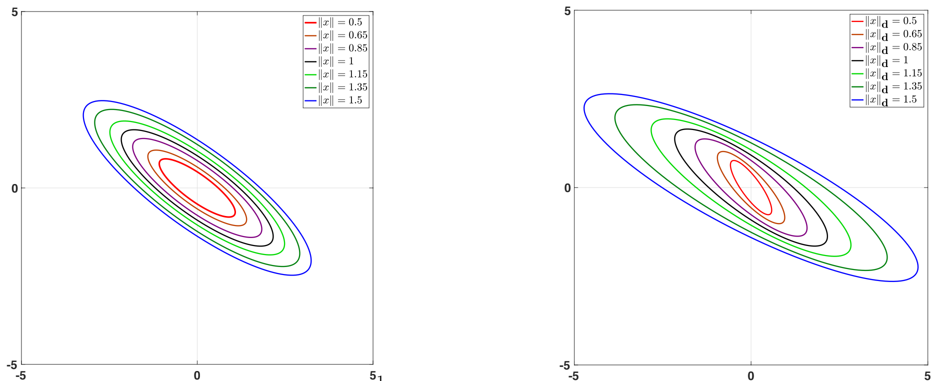
3) $\|x\tilde{+}y\|_{\mathbf{d}}\le\|x\|_{\mathbf{d}}+\|y\|_{\mathbf{d}}$

where $\tilde{+}$ is an alternative vector addition in $\mathbb{R}^n$ satisfying all axiomes of the vector space together with the scalar multiplication defined as $\alpha\tilde{\cdot}x=\text{sign}(\alpha)\mathbf{d}(-\ln|\alpha|)x$.

Canonical[8]homogeneous norm for monotone dilations

$$\|x\|_{\mathbf{d}}=e^{s_x} \quad \text{where} \quad s_x\in\mathbb{R}:\|\mathbf{d}(-s_x)x\|=1 \quad x\neq\mathbf{0}$$

[8] A homogeneous norm induced by a canonical norm of the space.



$\|x\| = (0.5x_1^2 + 0.4x_1x_2 + x_2^2)^{\frac{1}{2}}$ $\quad$ $\|x\|_{\mathbf{d}}$ induced by $\|x\|$ and $\mathbf{d}(s) = \mathrm{diag}\{e^{2s}, e^s\}$

Figure 2.2: Level sets of the weighted Euclidean norm and the **d**-homogeneous norm.

Lemma (*Polyakov 2019* [14])

If a linear continuous dilation $\mathbf{d}$ in $\mathbb{R}^n$ is strictly monotone with respect to the norm $\|x\| = \sqrt{x^\top P x}, P \succ 0$ then

1) $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \mapsto \mathbb{R}_+$ is single-valued and $\|\cdot\|_{\mathbf{d}} \in C^\infty(\mathbb{R}^n \backslash \{\mathbf{0}\})$;
2) $\|x\|_{\mathbf{d}} \to 0$ as $x \to \mathbf{0}$ and $x \neq \mathbf{0} \Leftrightarrow \|x\|_{\mathbf{d}} \neq \mathbf{0}$;
3) $\|\pm \mathbf{d}(s)x\|_{\mathbf{d}} = e^s \|x\|_{\mathbf{d}}, \forall s \in \mathbb{R}, \forall x \in \mathbb{R}^n$;
4) $\|x\| = 1 \Leftrightarrow \|x\|_{\mathbf{d}} = 1, \quad \|x\| < 1 \Leftrightarrow \|x\|_{\mathbf{d}} < 1$ and $\|x\| > 1 \Leftrightarrow \|x\|_{\mathbf{d}} > 1$;
5) for $0 < 2\beta = \lambda_{\min}\left(PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P\right) \leq \lambda_{\max}\left(PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P\right) = 2\alpha$ one has

$$|\|x_1\|_{\mathbf{d}}^{\alpha} - \|x_2\|_{\mathbf{d}}^{\alpha}| \leq \|x_1 - x_2\|, \quad \forall x_1, x_2 \in B, \tag{2.5}$$

$$\left|\|x_1\|_{\mathbf{d}}^{\beta} - \|x_2\|_{\mathbf{d}}^{\beta}\right| \leq \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^n \backslash B, \tag{2.6}$$

where $B = \{x :\in \mathbb{R}^n : \|x\| \leq 1\}$ is the unit ball in $\mathbb{R}^n$.

6) the derivative of $\|\cdot\|_{\mathbf{d}}$ is given by

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \|x\|_{\mathbf{d}} \frac{x^\top \mathbf{d}^\top(-\ln \|x\|_{\mathbf{d}}) P \mathbf{d}(-\ln \|x\|_{\mathbf{d}})}{x^\top \mathbf{d}^\top(-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x}, \quad x \neq \mathbf{0}. \tag{2.7}$$

The canonical homogeneous norm introduces an alternative norm topology in $\mathbb{R}^n$.

Generalized homogeneous homeomorphism on $\mathbb{R}^n$

Let a linear continuous dilation $\mathbf{d}$ be strictly monotone with respect to the norm $\|x\| = \sqrt{x^\top P x}$. The mapping $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ given by

$$\Phi(x) = \|x\|_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x, \quad x \in \mathbb{R}^n \tag{2.8}$$

is a homeomorphism in $\mathbb{R}^n$, its inverse has the form

$$\Phi^{-1}(z) = \|z\|^{-1} \mathbf{d}(\ln \|z\|)z, \quad z \in \mathbb{R}^n.$$

with $\Phi(\mathbf{0}) = \Phi^{-1}(\mathbf{0}) = \mathbf{0}$ by continuity.

The following theorem justifies the name "norm" for the functional $\|\cdot\|_{\mathbf{d}}$

Theorem (*Polyakov 2020* [15])

Let a linear dilation $\mathbf{d}$ be strictly monotone with respect to the norm $\|x\| = \sqrt{x^\top P x}$. Let an addition of vectors $\tilde{+} : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ and a multiplication by a scalar $\tilde{\cdot} : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ be defined as follows

- $x \tilde{+} y := \Phi^{-1}(\Phi(x) + \Phi(y))$, where $x, y \in \mathbb{R}^n$,
- $\lambda \tilde{\cdot} x := \mathrm{sign}(\lambda)\mathbf{d}(\ln|\lambda|)x$, where $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$.

Then the set $\mathbb{R}^n$ together with the operations $\tilde{+}$ and $\tilde{\cdot}$ is a linear vector space $\mathbb{R}^n_{\mathbf{d}}$ with the norm $\|\cdot\|_{\mathbf{d}}$.

2.4 Homogeneous Systems

2.4.1 Homogeneous Mappings

Homogeneous function (*Kawski 1991* [6])

A function $h : \mathbb{R}^n \mapsto \mathbb{R}$ is said to be **d**-homogeneous of a degree $\nu \in \mathbb{R}$ if

$$h(\mathbf{d}(s)x) = e^{\nu s} h(x) \quad \text{for} \quad s \in \mathbb{R}, \quad x \in \mathbb{R}^n, \tag{2.9}$$

where $\mathbf{d}$ is a dilation in $\mathbb{R}^n$.

Let $\mathcal{H}_{\mathbf{d}}(\mathbb{R}^n)$ be a set of all **d**-homogeneous functions $\mathbb{R}^n \to \mathbb{R}$ and $\deg_{\mathbf{d}}(h) \in \mathbb{R}$ denote the homogeneity degree of $h \in \mathcal{H}_{\mathbf{d}}(\mathbb{R}^n)$.

Elements homogeneous arithmetics for functions $h, g \in \mathcal{H}_{\mathbf{d}}(\mathbb{R}^n)$:

1. $\alpha h \in \mathcal{H}_{\mathbf{d}}(\mathbb{R}^n)$ and $\deg_{\mathbf{d}}(\alpha h) = \deg_{\mathbf{d}}(h)$ for any $\alpha \in \mathbb{R}$;
2. $h + g \in \mathcal{H}_{\mathbf{d}}(\mathbb{R}^n)$ provided that $\deg_{\mathbf{d}}(h) = \deg_{\mathbf{d}}(g)$;
3. $h \cdot g \in \mathcal{H}_{\mathbf{d}}(\mathbb{R}^n)$ and $\deg_{\mathbf{d}}(h \cdot g) = \deg_{\mathbf{d}}(h) + \deg_{\mathbf{d}}(g)$;
4. $\frac{h}{g} \in \mathcal{H}_{\mathbf{d}}(\mathbb{R}^n)$ and $\deg_{\mathbf{d}}\left(\frac{h}{g}\right) = \deg_{\mathbf{d}}(h) - \deg_{\mathbf{d}}(g)$ if $g(x) \neq \mathbf{0}$, $\forall x \in S$;
5. if $h(x) = c$ for all $x \in \mathbb{R}^n$ then $h \in \mathcal{H}_{\mathbf{d}}(\mathbb{R}^n)$ and $\deg_{\mathbf{d}}(h) = 0$ for $c \in \mathbb{R}\backslash\{0\}$. If $c = 0$ then $\deg_{\mathbf{d}}(h)$ is any.

Properties of homogeneous functions (*Bhat & Bernstein 2005* [2])

Let $h \in \mathcal{H}_{\mathbf{d}}(\mathbb{R}^n)$ be such that $\sup_{x \in S} |h(x)| < +\infty$.

- If $\deg_{\mathbf{d}}(h) > 0$ then h is bounded on any **d**-homogeneous ball $B_{\mathbf{d}}(r)$ and $h(x) \to 0$ as $x \to 0$
- If $\deg_{\mathbf{d}}(h) < 0$ then h is bounded on any set $\mathbb{R}^n \backslash B_{\mathbf{d}}(r)$ with $r > 0$ and $h(x) \to 0$ as $x \to \infty$
- If $\deg_{\mathbf{d}}(h) = 0$ then h is uniformly bounded on $\mathbb{R}^n$ and, moreover, $h \equiv \text{const}$ provided that h is continuous at zero.

Euler's homogeneous function theorem (*Polyakov 2020* [15])

If $h \in \mathcal{H}_{\mathbf{d}}(\mathbb{R}^n)$ is differentiable on $\mathbb{R}^n \backslash \{\mathbf{0}\}$ then

$$\frac{\partial h(x)}{\partial x} G_{\mathbf{d}} x = \deg_{\mathbf{d}}(h) \cdot h(x), \quad \forall x \neq \mathbf{0}, \tag{2.10}$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ is a generator of the linear dilation $\mathbf{d}$.

Homogeneous vector field (*Kawski 1991* [6])

A <u>vector field</u> $f:\mathbb{R}^n \mapsto \mathbb{R}^n$ is said to be $\mathbf{d}$-homogeneous of degree $\mu \in \mathbb{R}$ if

$$f(\mathbf{d}(s)x) = e^{\mu s}\mathbf{d}(s)f(x) \quad \text{for all} \quad s \in \mathbb{R}, \quad u \in \mathbb{R}^n, \tag{2.11}$$

where $\mathbf{d}$ is a dilation in $\mathbb{R}^n$.

Let $\mathcal{F}_{\mathbf{d}}(\mathbb{R}^n)$ be a set of $\mathbf{d}$-homogeneous vector fields $\mathbb{R}^n \mapsto \mathbb{R}^n$ and $\deg_{\mathbf{d}}(f)$ denote the homogeneity degree.

Elements homogeneous arithmetics for vector fields:

- If $h \in \mathcal{H}_{\mathbf{d}}(\mathbb{R}^n)$ and $f \in \mathcal{F}_{\mathbf{d}}(\mathbb{R}^n)$ then
  $$h \cdot f \in \mathcal{F}_{\mathbf{d}}(\mathbb{R}^n) \text{ and } \deg_{\mathbf{d}}(h \cdot f) = \deg_{\mathbf{d}}(h) + \deg_{\mathbf{d}}(f)$$
- If $f_1, f_2 \in \mathcal{F}_{\mathbf{d}}(\mathbb{R}^n)$ and $\deg_{\mathbf{d}}(f_1) = \deg_{\mathbf{d}}(f_2)$ then
  $$f_1 + f_2 \in \mathcal{F}_{\mathbf{d}}(\mathbb{R}^n) \text{ and } \deg_{\mathbf{d}}(f_1 + f_2) = \deg_{\mathbf{d}}(f_1) = \deg_{\mathbf{d}}(f_2)$$
- If $f_1, f_2 \in \mathcal{F}_{\mathbf{d}}(\mathbb{R}^n)$ then
  $$f_1(f_2) \in \mathcal{F}_{\mathbf{d}}(\mathbb{R}^n)$$
  provided that $\deg_{\mathbf{d}}(f_2) = 0$ or f_1 is standard homogeneous.

Properties of homogeneous vector fields (*Polyakov 2020* [15])

Let $f \in \mathcal{F}_{\mathbf{d}}(\mathbb{R}^n)$ be such that $M := \sup_{x \in S} \|f(x)\| < +\infty$

- If $\deg_{\mathbf{d}}(f) + \beta > 0$ then f is bounded on $B_{\mathbf{d}}(r)$ and $\|f(x)\| \to 0$ as $\|x\| \to 0$
- If $\deg_{\mathbf{d}}(f) + \alpha < 0$ then f is bounded on $\mathbb{R}^n \backslash B_{\mathbf{d}}(r)$ and $\|f(x)\| \to 0$ as $\|x\| \to +\infty$
- If $\deg_{\mathbf{d}}(f) + \beta = \deg_{\mathbf{d}}(f) + \alpha = 0$ then f is uniformly bounded on $\mathbb{R}^n$,

where $\alpha = \lambda_{\max}\left(P^{1/2}G_{\mathbf{d}}P^{-1/2} + P^{-1/2}G_{\mathbf{d}}P^{1/2}\right)$ and $\beta = \lambda_{\min}\left(P^{1/2}G_{\mathbf{d}}P^{-1/2} + P^{-1/2}G_{\mathbf{d}}P^{1/2}\right)$.

Euler's theorem for vector fields (*Polyakov 2020* [15])

If $f \in \mathcal{F}_{\mathbf{d}}(\mathbb{R}^n)$ is differentiable on $\mathbb{R}^n \backslash \{\mathbf{0}\}$ then

$$\frac{\partial f(x)}{\partial x} G_{\mathbf{d}} x = (\deg_{\mathbf{d}}(f) I_n + G_{\mathbf{d}}) f(x) \text{ for all } x \in \mathbb{R}^n \backslash \{\mathbf{0}\}. \tag{2.12}$$

On linear homogeneous vector fields (*Polyakov 2020, Zimenko et al. 2020*)

The following three claims are equivalent ([15], [23]):

1) a vector field $x \mapsto Ax$ with $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is $\mathbf{d}$-homogeneous of degree $\mu \neq 0$ with respect to a linear continuous dilation $\mathbf{d}$ in $\mathbb{R}^n$.

2) there exists an anti-Hurwitz matrix $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ such that

$$AG_{\mathbf{d}} = (\mu I + G_{\mathbf{d}})A; \tag{2.13}$$

3) the matrix A is nipotent.

Conclusion:

- $A^n \neq \mathbf{0} \quad \Rightarrow \quad G_{\mathbf{d}} = I_n,\ \mu = 0$

- $A^n = \mathbf{0} \quad \Rightarrow \quad \forall \mu \neq \mathbf{0},\ \exists G_{\mathbf{d}}$ – anti-Hurwitz : $AG_{\mathbf{d}} = (\mu I_n + G_{\mathbf{d}})A$

How to find $G_{\mathbf{d}}$ for a given $\mu \neq 0$?

A possible solution:

1. find G_0 such that $AG_0 = (I_n + G_0)A$ (i.e., solve the linear equation)
2. take $G_{\mathbf{d}} = \epsilon I_n + \mu G_0$ with a large enough $\epsilon > 0$ (to make $G_{\mathbf{d}}$ anti-Hurwitz)

Local Homogeneity (*Andrieu et al 2008* [1])

A $\mathbf{d}$-homogeneous vector field $f_L : \mathbb{R}^n \mapsto \mathbb{R}^n$ of degree $\nu \in \mathbb{R}$ is said be $\mathbf{d}$-homogeneous approximation of f at L-limit (with $L = 0$ or $L = \infty$) if

$$\lim_{r\to L^+} \sup_{x\in S} \|r^{-\nu}\mathbf{d}(-\ln r) f(\mathbf{d}(\ln r)x) - f_L(x)\| = 0,$$

where $S = \{x :\in \mathbb{R}^n : \|x\| = 1\}$ is the unit sphere in $\mathbb{R}$.

Example: If $f(x) = -x^3 + x^4 - x^5$ then

- the linearization at zero gives $\left.\frac{\partial f(z)}{\partial z}\right|_{z=0} x = \left.3z^2\right|_{z=0} x = 0$
- the homogeneous approximation at zero gives $f_0(x) = -x^3$.

2.4.2 Homogeneous Differential Equations

Any linear system

$$\dot{x} = Ax, \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n\times n}, \quad x(0) = x_0$$

is standard homogeneous and its solution $x(t, x_0) = e^{tA}x_0$ is symmetric with respect to the scaling of the initial condition $x(t, e^s x_0) = e^s x(t, x_0)$

Symmetry of a flow of generalized homogeneous linear vector field

Let $\mathbf{d}$ be a linear dilation and let a linear vector field $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ given by

$$f(x) = Ax, \quad x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n\times n}$$

be $\mathbf{d}$-homogeneous of degree $\mu \in \mathbb{R}^n$ (i.e., $A\mathbf{d}(s) = e^{\mu s}\mathbf{d}(s)A$) then

$$e^{tA}\mathbf{d}(s) = \mathbf{d}(s)e^{tAe^{\mu s}}, \quad \forall s \in \mathbb{R}, \quad \forall t \in \mathbb{R}.$$

Let $x(t, x_0)$ denote a solution of the system

$$\dot{x}(t) = f(x(t)), \quad t > 0, \tag{2.14}$$

with the initial condition $x(0) = x_0 \in \mathbb{R}^n$.

Theorem (*Zubov 1958, Kawski 1991*)

Let $\mathbf{d}$ be a dilation in $\mathbb{R}^n$ and the vector field f be $\mathbf{d}$-homogeneous of degree $\mu \in \mathbb{R}$. If $x(t, x_0)$ with $t \in [0, T]$ is a solution of the system (2.14) with the initial condition $x(0) = x_0$ then, for any $s \in \mathbb{R}$,

$$x(t, \mathbf{d}(s)x_0) = \mathbf{d}(s)x(e^{\mu s}t, x_0), \quad t \in [0, e^{-\mu s}T]$$

is a solution of (2.14) with the scaled initial condition $x(0) = \mathbf{d}(s)x_0$.

Corollary

Let f be a continuous $\mathbf{d}$-homogeneous vector field of degree $\mu \leq 0$ then the system $\dot{x} = f(x)$ is complete[9].

[9] A solution x of the system (2.14) has a finite-time blow-up if there exists $T \in \mathbb{R}$ such that $|x(t)| \to +\infty$ as $t \to T$.
A system is *complete* if all its solutions have no finite-time blow up (in other words, if any solution exists for all $t \in \mathbb{R}$)

Lyapunov Stability (*Lyapunov 1892*)

The system (2.14) is said to be locally (globally) uniformly *Lyapunov stable* if[10] $\exists \varepsilon \in \mathcal{K}$

$$|x(t, x_0)| \leq \varepsilon(|x_0|), \quad \forall t \geq 0, \quad \forall x_0 \in \Omega,$$

where Ω is a neighborhood of the origin (resp., $\Omega = \mathbb{R}^n$)

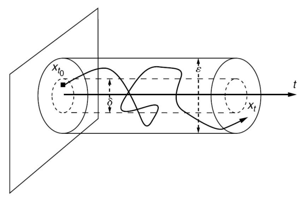


Figure 2.3: Illustration of Lyapunov stability

Proposition (*Bhat & Bernstein 2005* [2])

Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a continuous **d**-homogeneous vector field. The system (2.14) is Lyapunov stable if and only if it has a positively invariant[11] bounded neighborhood of the origin.

Asymptotic Stability (*Lyapunov 1892*)

The system (2.14) is said to be locally (globally) uniformly *asymptotically stable* if[12] $\exists \beta \in \mathcal{KL}$

$$\|x(t, x_0)\| \leq \beta(|x_0|, t), \quad \forall t \geq 0, \quad \forall x_0 \in \Omega,$$

where Ω is a neighborhood of the origin (resp., $\Omega = \mathbb{R}^n$)

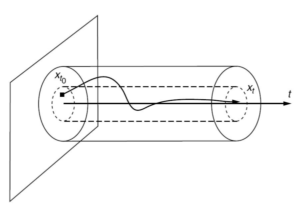


Figure 2.4: Illustration of asymptotic stability

[10] A function $\varepsilon : [0, +\infty) \mapsto [0, +\infty)$ is said to be of the class $\mathcal{K}$ if ε is continuous, strictly increasing and $\varepsilon(0) = 0$.

[11] A set $\Omega \subset \mathbb{R}^n$ is positively invariant if $x_0 \in \Omega \Rightarrow x(t, x_0) \in \Omega, \forall t \geq 0$.

[12] A function $\beta : [0, +\infty) \times [0, +\infty)$ is said to be of the class $\mathcal{KL}$ if $\beta(\cdot, s) \in \mathcal{K}$ for any fixed $s \geq 0$ and $\beta(r, \cdot)$ is strictly decreasing to zero for any fixed $r \geq 0$.

Zubov-Rosier Theorem (*Zubov 1958* [24] and *Rosier 1992*[20])

Let a continuous vector field $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be $\mathbf{d}$-homogeneous of degree μ. Let $m > 0$ be an arbitrary positive number.

The system (2.14) is asymptotically stable if and only if there exists a positive definite homogeneous function $V \in C(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \backslash \mathbf{0})$ of degree m:

$$\dot{V}(x) \leq -\rho V^{1+\frac{\mu}{m}}(x), \quad \forall x \in \mathbb{R}^n \backslash \{\mathbf{0}\}, \tag{2.15}$$

where $\rho > 0$ is some number.

Corollary (*Nakamura et al 2002* [9]

Let f be $\mathbf{d}$-homogeneous of degree $\mu \in \mathbb{R}$. If (2.14) is locally asymptotically stable then it is

- globally uniformly **finite-time stable**[13] for $\mu < 0$ with a settling-time function T being continuous at $x = 0$;
- globally **nearly fixed-time stable** for $\mu > 0$, i.e.

$$\forall r > 0, \exists T_r > 0 \text{ such that } \|x(t, x_0)\| \leq r, \forall t \geq T_r, \forall x_0 \in \mathbb{R}^n.$$

Let us consider the perturbed nonlinear system:

$$\dot{x} = f(x, q), \quad t > 0, \quad x(t) \in \mathbb{R}^n, \quad q(t) \in \mathbb{R}^k, \quad x(0) = x_0 \tag{2.16}$$

Definition (*Sontag 1989* [22])

A system (2.16) is said to be Input-to-State Stable (ISS) with respect to $q \in L^\infty(\mathbb{R}, \mathbb{R}^k)$ if there exist[14] $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\|q\|_{L^\infty((t_0,t),\mathbb{R}^k)}). \tag{2.17}$$

Theorem (*Hong 2001* [4])

Let $\mathbf{d}_q$ be a dilation in $\mathbb{R}^n$ and $\mathbf{d}_q$ be a dilation in $\mathbb{R}^m$ such that $\exists \mu \in \mathbb{R}$:

$$f(\mathbf{d}_x(s)x, \mathbf{d}_q(s)q) = e^{\mu s}\mathbf{d}_x(s)f(x, q) \quad \forall x \in \mathbb{R}^m, \forall q \in \mathbb{R}^m, \forall s \in \mathbb{R}.$$

If the system $\dot{x} = f(x, \mathbf{0})$ is asymptotically stable then the system (2.16) is ISS.

[13] A system is said to be globally uniformly finite-time stable if it is Lyapunov stable and there exists a locally settling-time function $T : \mathbb{R}^n \mapsto [0, +\infty)$ such that

$$x(t, x_0) = \mathbf{0}, \quad \forall t \geq T(x_0), \quad \forall x_0 \in \mathbb{R}^n.$$

2.4.3 Homogeneous Control Design

Let us consider the control system

$$\dot{x} = Ax + Bu, \quad t > 0, \tag{2.18}$$

where $x \in \mathbb{R}^n$ - system state, $u \in \mathbb{R}^m$ - control input, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$.

The Problem of Homogeneous Stabilization

Given $\mu \in \mathbb{R}$ the control aim is to design a dilation $\mathbf{d}$ in $\mathbb{R}^n$ and a feedback control $u : \mathbb{R}^n \mapsto \mathbb{R}^m$ such that the closed-loop system is

1. $\mathbf{d}$-homogeneous of degree μ, i.e. for $f(x) = Ax + Bu(x)$ one holds
$$f(\mathbf{d}(s)x) = e^{\mu s}\mathbf{d}(s)f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R};$$
2. asymptotically stable ($\Rightarrow$ finite-time stability for $\mu < 0$).

Theorem 1 (inspired by [17, 18, 23, 15, 10])

The system (2.18) is homogeneously stabilizable with $\mu \neq 0$ **if and only if** the pair $\{A, B\}$ is controllable[15]. For any controllable pair $\{A, B\}$ one holds

1) the linear algebraic equation
$$AG_0 - G_0A + BY_0 = A, \quad G_0B = \mathbf{0} \tag{2.19}$$
has a solution $Y_0 \in \mathbb{R}^{m\times n}$, $G_0 \in \mathbb{R}^{n\times n}$ and for any solution one hold
   - the matrix $G_0 - I_n$ is invertible is invertible;
   - $G_{\mathbf{d}} = I_n + \mu G_0$ is anti-Hurwitz for $\mu \leq 1/\tilde{n}$, where $\tilde{n} \in \mathbb{N}$ is a minimal number such that $\mathrm{rank}[B, AB, ..., A^{\tilde{n}-1}B] = n$;
   - the matrix $A_0 = A + BK_0$ is nilpotent, $K_0 = Y_0(G_0 - I_n)^{-1}$ and
$$A_0G_{\mathbf{d}} = (G_{\mathbf{d}} + \mu I_n)A_0, \quad G_{\mathbf{d}}B = B; \tag{2.20}$$

2) the linear algebraic system
$$\begin{gathered} A_0X + XA_0^\top + BY + Y^\top B^\top + \rho(G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top) = \mathbf{0}, \\ G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top \succ 0, \quad X = X^\top \succ 0 \end{gathered} \tag{2.21}$$
has a solution $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{m\times n}$ for any $\rho \in \mathbb{R}_+$;

3) the homogeneous norm $\|\cdot\|_{\mathbf{d}}$ induced by $\|x\| = \sqrt{x^\top X^{-1}x}$ is a Lyapunov function of the system (2.18) with the feedback law
$$u(x) = K_0x + \|x\|_{\mathbf{d}}^{1+\mu}K\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x, \quad K = YX^{-1}, \tag{2.22}$$
where $\mathbf{d}$ is a dilation generated by $G_{\mathbf{d}}$; moreover,
$$\tfrac{d}{dt}\|x\|_{\mathbf{d}} = -\rho\|x\|_{\mathbf{d}}^{1+\mu}, \quad x \neq \mathbf{0}; \tag{2.23}$$

4) $u \in C^\infty(\mathbb{R}^n\backslash\{\mathbf{0}\})$ and $|u(x)| \leq K_0|x| + \lambda_{\max}(X)\|x\|_{\mathbf{d}}^{1+\mu}$, $\forall x \in \mathbb{R}^n$, $\forall \mu \geq -1$;

5) the system (2.18), (2.22) is $\mathbf{d}$-homogeneous of degree μ.

2.4.4 Homogeneous Observer Design

Let us consider the system

$$\dot{x} = Ax + p(t), \quad t > 0, \quad y = Cx \tag{2.24}$$

where $x \in \mathbb{R}^n$ - (unknown) system state, $p(t) \in \mathbb{R}^n$ - known exogenous input, $y \in \mathbb{R}^k$ - measured system output, $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{k\times n}$.

The Problem of Homogeneous Observer Design

Given $\mu \in \mathbb{R}$ we need to design a dilation $\mathbf{d}$ in $\mathbb{R}^n$ and an observer:

$$\dot{z} = Az + p + g(Cz - y), \quad z(0) = \mathbf{0} \quad g : \mathbb{R}^k \mapsto \mathbb{R}^n$$

such that the error equation

$$\dot{\epsilon} = A\epsilon + g(C\epsilon), \quad \epsilon = z - x$$

1. is $\mathbf{d}$-homogeneous of degree μ, i.e. for $f(\epsilon) = A\epsilon + g(C\epsilon)$ one holds
$$f(\mathbf{d}(s)\epsilon) = e^{\mu s}\mathbf{d}(s)f(\epsilon), \quad \forall \epsilon \in \mathbb{R}^n, \quad \forall s \in \mathbb{R};$$
2. is asymptotically stable.

Theorem 2 (inspired by [2, 7, 15, 10])

The system (2.24) is homogeneously observable with degree $\mu \neq 0$ **if and only if** the pair $\{A, C\}$ is observable. For any observable pair $\{A, C\}$ one holds

1) the linear algebraic equation
$$AG_0 - G_0A + Y_0C = A, \quad CG_0 = \mathbf{0} \tag{2.25}$$
has a solution $Y_0 \in \mathbb{R}^{n\times k}$, $G_0 \in \mathbb{R}^{n\times n}$ and for any solution one holds
   - the matrix $G_0 + I_n$ is invertible
   - the matrix $G_{\mathbf{d}} = I_n + \nu G_0$ is anti-Hurwitz for $\nu \geq -1/\tilde{n}$, where $\tilde{n}$ is a minimal natural number such that
   $$\text{rank}\begin{bmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{bmatrix} = n.$$
   - the matrix $A_0 = A + L_0C$ is nilpotent, $L_0 = -(G_0 + I_n)^{-1}Y_0$ and
   $$A_0G_{\mathbf{d}} = (\nu I_n + G_{\mathbf{d}})A_0, \quad CG_{\mathbf{d}} = C;$$
2) the algebraic system
$$\begin{gathered} PA_0 + A_0^\top P + YC + C^\top Y^\top + \rho(PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P) = \mathbf{0}, \\ PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, \quad P = P^\top \succ 0 \end{gathered} \tag{2.26}$$
has a solution $P \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{k\times n}$ for any $\rho \in \mathbb{R}_+$.

[15] A pair $\{A, B\}$ is controllable if and only if $\text{rank}[B, AB, ..., A^{n-1}B] = n$.

3) the canonical homogeneous norm $\|\epsilon\|_{\mathbf{d}}$ induced by the weighted Euclidean norm $\|\epsilon\| = \sqrt{\epsilon^{\top} P \epsilon}$ is a Lyapunov function (at least if ν close to 0) for the error equation (2.28) of the observer

$$\dot{z} = Az + p + \left(L_0 + |Cz-y|^{\nu-1}\mathbf{d}(\ln|Cz-y|)L\right)(Cz-y), \quad z(0) = \mathbf{0}, \quad (2.27)$$

where $L = P^{-1}Y$;

- the error equation

$$\dot{\epsilon} = (A_0 + |C\epsilon|^{\nu-1}\mathbf{d}(\ln|C\epsilon|)LC)\epsilon, \quad \epsilon = z - x \quad (2.28)$$

is continuous for $\nu > -1/\tilde{n}$ and discontinuous for $\nu = -1/\tilde{n}$;

- the error system (2.27) is $\mathbf{d}$-homogeneous of degree ν,

Remarks

- without loss of generality, the identity $= 0$ in the system of LMIs (2.21) (resp. (2.26)) can replaced with inequality $\preceq 0$;
- combing the homogeneous controllers/observers with positive and negative degrees, a locally homogeneous **fixed-time stable** system can be designed:

$$\exists T_{\max} > 0 \quad : \quad x(t, x_0) = \mathbf{0}, \quad \forall t \geq T_{\max}, \quad \forall x_0 \in \mathbb{R}^n$$

$$(\text{resp., } \exists T_{\max} > 0 \quad : \quad z(t) = x(t, x_0), \quad \forall t \geq T_{\max}, \quad \forall x_0 \in \mathbb{R}^n)$$

2.5 On Discretization of Homogeneous Systems

2.5.1 Control Discretization

Due to a digital implementation of controller one holds

$$u(t) = u_j, \quad \forall t \in [t_j, t_{j+1}), j = 0, 1, 2, ...$$

where $t_{j+1} - t_j \geq h > 0$, $t_0 = 0$. Moreover, the measurements are samples as well. Let us assume (for simplicity) that the measurements and control samplings are synchronized, i.e. at time $t = t_j$ we can measure $x_j = x(t_j)$. Several algorithms of digital implementation of homogeneous controller can be suggested in this case [11].

Explicit discretization of homogeneous control

$$u_j = K_0 x_j + \|x_j\|_{\mathbf{d}}^{1+\mu} K\mathbf{d}(-\ln\|x_j\|_{\mathbf{d}})x_j$$

Using the so-called semi implicit Euler method the closed-loop system can be approximated as follows

$$x(t_{j+1}) \approx \tilde{x}_{j+1} = x_j + h(A + BK_j)\tilde{x}_{j+1}, \quad K_j = K_0 + \|x_j\|^{1+\mu} K\mathbf{d}(-\ln\|x_j\|_{\mathbf{d}})$$

Hence, $\tilde{x}_{j+1} = (I_n - h(A+BK_j)^{-1}x_j$ the semi-implicit control discritization can be defined as follows.

Semi-implicit discretization of homogeneous control

$$u_j = K_j(I_n-(t_{j+1}-t_j)(A+BK_j))^{-1}x_j, \quad K_j = K_0+\|x_j\|_{\mathbf{d}}^{1+\mu}K\mathbf{d}(-\ln\|x_j\|_{\mathbf{d}})$$

The methodology of consistent discretization of homogeneous systems are developed in [16], [19].

2.5.2 Observer Discretization

To implement a homogeneous observer in a digital device the system (2.27) has to be properly discretized under assumption that the output measurements y and the exogenous input p are sampled:

$$y(t) = y(t_j), \quad p(t) = p(t_j), \quad t \in [t_j, t_{j+1}), \quad j = 0,1,2,...$$

where $t_{j+1} - t_j \geq h > 0$. The observer's discretizations are defined using the same ideas as controllers discretization.

Explicit discretization of homogeneous observer

$$z_{j+1} = z_j+(t_{j+1-t_j})\left(Az_j+p_j+(L_0+|Cz_j-y_j|^{\nu-1}\mathbf{d}(\ln|Cz_j-y_j|)L)(Cz_j-y_j)\right)$$

where $z_0 = z(0)$ and $z_j \approx z(t_j)$. The semi-implicit Euler's method gives

$$z_{j+1} = z_j + h\left(Az_{j+1} + p_j + (L_0 + |Cz_j - y_j|^{\nu-1}\mathbf{d}(\ln|Cz_j - y_j|)L(Cz_{j+1} - y_j)\right)$$

Semi-implicit discretization of homogeneous observer

$$z_{j+1} = \tilde{A}_j^{-1}\left(z_j + (t_{j+1} - t_j)\left(p_j - L_j y_j\right)\right)$$

where $\tilde{A}_j = I_n - h(A + L_jC)$ and

$$L_j = L_0 + |Cz_j - y_j|^{\nu-1}\mathbf{d}(\ln|Cz_j - y_j|)L.$$

The presented discretizations have different computational complexity and can be selected for practical implementation dependently of the available computational resources.

2.6 Theoretical conclusions

- Homogeneous systems may have faster convergence, better robustness and smaller overshoots than linear systems.
- Theorems 1 and 2 are constructive and provide a way to define parameters of homogeneous controllers/ observer by solving certain algebraic systems. All functions of controllers and observers design are developed in HCS Toolbox based on the mentioned theorems and their corollaries.
- In the view of the structure of the homogeneous controllers/observers many existing linear controllers/observers can be easily upgraded/transformed to homogeneous ones.

Chapter 3

Homogeneous Systems in MATLAB

3.1 Controllers

This section surveys functions of HCS Toolbox for homogeneous control design and implementation in MATLAB. It is essentially based on the concept of the homogeneous norm considered in Chapter 2.

3.1.1 Homogeneous Proportional Control (HPC)

Model of the control system:

$$\dot{x} = Ax + B(u + \gamma(t,x)), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad A \in \mathbb{R}^{n\times m}, \quad B \in \mathbb{R}^{n\times m}$$

where the pair $\{A, B\}$ is controllable and $\gamma : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^m$ is an unknown function.

Homogeneous control:

$$u_{hpc} = K_0 x + \|x\|_{\mathbf{d}}^{1+\mu} K \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x, \quad K_0 \in \mathbb{R}^{m\times n}, \quad K \in \mathbb{R}^{m\times n}$$

where $\mu \geq -1$, $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$ is a dilation in $\mathbb{R}^n$, $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ and the homogeneous norm $\|x\|_{\mathbf{d}}$ is induced by the weighted Euclidean norm $\|x\| = \sqrt{x^\top P x}$ in $\mathbb{R}^n$, $P \in \mathbb{R}^{n\times n}$.

Properties:

- finite-time stabilization for negative homogeneity degree $\mu < 0$

$$x(t) = 0, \quad \forall t \geq \|x(0)\|_{\mathbf{d}}^{-\mu}/(-\mu\rho);$$

- nearly fixed-time stabilization for positive homogeneity degree $\mu > 0$

$$\|x(t)\|_{\mathbf{d}} \leq r, \quad \forall t \geq \frac{1}{\mu\rho r^{\mu}}, \quad \forall r > 0, \quad \forall x(0) \in \mathbb{R}^n;$$

- **d**-homogeneity of the unperturbed system (if $\gamma = \mathbf{0}$) $\Rightarrow$ ISS with respect to measurement noises in x and additive perturbations in the model;

- rejection of the matched perturbation γ if

$$|\gamma(t,x)| \leq \gamma_{\max}\|x\|_{\mathbf{d}}^{1+\mu}.$$

HPC Design

The function `hpc_design` computes parameters K_0, K, G_d and P of HPC for given A, B, $\mu \geq -1$, $\rho > 0$ and $\gamma_{\max} \geq 0$

- **Input parameters**: A, B, μ and ρ (by default $\rho = 1$) and $\gamma_{\max}$ (by default $\gamma_{\max} = 0$)

- **Output parameters**: K_0, K, G_d and P

HPC Implementation

- The function `e_hpc` computes *explicit* discretization of u_{hpc}
  - **Input parameters**: x, K_0, K, μ, $G_{\mathbf{d}}$, P
  - **Output parameters**: u_{hpc}

- The function `si_hpc` computes *semi-implicit* discretization of u_{hpc}
  - **Input parameters**: h (sampling period), x, A, B, K_0, K, μ, $G_{\mathbf{d}}, P$
  - **Output parameters**: u_{hpc}

- The function `c_hpc` computes *consistent* discretization of u_{hpc} if $\gamma_{max} = 0$
  - **Input parameters**: h (sampling period), x, A, B, K_0, K, μ, $G_{\mathbf{d}}, P$, ρ
  - **Output parameters**: u_{hpc}

Use[1] `demo_hpc.m` from `HCS Toolbox` as a demo of HPC design

[1] To open a demo please type 'edit <name_of_demo>.m' in the Command Line of MATLAB

3.1.2 Fixed-time HPC

Model of the control system:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad A \in \mathbb{R}^{n\times m}, \quad B \in \mathbb{R}^{n\times m}$$

where the pair $\{A, B\}$ is controllable.

Control law:

$$u_{fhpc} = K_0 x + \begin{cases} \|x\|_{\mathbf{d}_1}^{1+\mu_1} K\mathbf{d}_1(-\ln \|x\|_{\mathbf{d}_1})x & \text{if} \quad x^\top Px \le 1 \\ \|x\|_{\mathbf{d}_2}^{1+\mu_2} K\mathbf{d}_2(-\ln \|x\|_{\mathbf{d}_2})x & \text{if} \quad x^\top Px > 1 \end{cases} \tag{3.1}$$

where $\mu_1 < 0 < \mu_2$, $\mathbf{d}_k(s) = e^{s(I_n + \mu_k G_0)}$ and $\|x\|_{\mathbf{d}_k}$ is induced by $\|x\| = \sqrt{x^\top Px}$, $k = 1, 2$

Properties:

- fixed-time stabilization of linear plant:

$$x(t) = \mathbf{0}, \quad \forall t \ge T_{\max}, \quad \forall x(0) \in \mathbb{R}^n, \quad T_{\max} = \frac{1}{-\mu_1\rho} + \frac{1}{\mu_2\rho}.$$

- local homogeneity of closed-loop system (of degree μ_1 at 0 and of degree μ_2 at ∞) $\Rightarrow$ ISS with respect to measurement noises in x and additive perturbations in the model;

Fixed-time HPC Design

The function `fhpc_design` computes parameters $K_0, K, G_0, P, \mu_1, \mu_2$ and $\rho > 0$ of Fixed-time HPC for given system matrix A and control matrix B.

- **Input parameters**: A and B
- **Output parameters**: $K_0, K, G_0, P, \mu_1, \mu_2$ and ρ

Fixed-time HPC Implementation

- The function `e_fhpc` computes *explicit* discretization of u_{fhpc}
  - **Input parameters**: x, K_0, K, μ_1, μ_2, G_0, P
  - **Output parameters**: u_{fhpc}
- The function `si_fhpc` computes *semi-implicit* discretization of u_{hpc}
  - **Input parameters**: h (sampling period), x, A,B, K_0, K, μ_1, μ_2, G_0, P
  - **Output parameters**: u_{fhpc}

Use `demo_fhpc.m` from `HCS Toolbox` as a demo of Fixed-time HPC design

3.1.3 Homogeneous Sliding Mode Control (HSMC)

Model of the control system:

$$\dot{x}=Ax+B(u+\gamma(t,x)),\quad y=Cx,\quad x\in\mathbb{R}^n,\quad u\in\mathbb{R}^m,\quad A\in\mathbb{R}^{n\times m},\quad B\in\mathbb{R}^{n\times m},\quad C\in\mathbb{R}^{p\times n}$$

where $y\in\mathbb{R}^p$ is a controllable output and $\gamma:\mathbb{R}\times\mathbb{R}^n\mapsto\mathbb{R}^m$ is an unknown uniformly bounded function.

Control law:

$$u_{hsmc}=K_0x+K\mathbf{d}(-\ln\|Cx\|_{\mathbf{d}})Cx,\quad K_0\in\mathbb{R}^{m\times n},\quad K\in\mathbb{R}^{m\times p},$$

where $\mathbf{d}(s)=e^{sG_{\mathbf{d}}}$ is a dilation in $\mathbb{R}^p$, $G_{\mathbf{d}}\in\mathbb{R}^{p\times p}$ and the homogeneous norm $\|x\|_{\mathbf{d}}$ is induced by the weighted Euclidean norm $\|x\|=\sqrt{x^\top Px}$ in $\mathbb{R}^p$, $P\in\mathbb{R}^{p\times p}$.

Properties:

- enforces sliding mode on the surface $Cx=0$ in a finite time
$$Cx(t)=0,\quad\forall t\geq\|Cx(0)\|_{\mathbf{d}}/\rho;$$
- the output dynamics $\dot{y}=C\dot{x}$ is $\mathbf{d}$-homogeneous of degree -1.
- rejection of the matched perturbation γ if
$$|\gamma(t,x)|\leq\gamma_{\max}.$$

HSMC Design

The function `hsmc_design` computes parameters K_0,K,G_d and P of HPC for given A, B, $\rho>0$ and $\gamma_{\max}\geq0$

- **Input parameters**: A, B, ρ (by default $\rho=1$) and $\gamma_{\max}$ (by default $\gamma_{\max}=0$)
- **Output parameters**: K_0,K,G_d and P

HSMC Implementation

- The function `e_smc` computes *explicit* discretization of u_{hsmc}
  - **Input parameters**: x, C, K_0,K, $G_{\mathbf{d}}$, P
  - **Output parameters**: u_{hsmc}
- The function `si_hpc` computes *semi-implicit* discretization of u_{hsmc}
  - **Input parameters**: h (sampling period), x, A,B, C, K_0,K, $G_{\mathbf{d}},P$
  - **Output parameters**: u_{hsmc}

Use `demo_hsmc.m` from `HCS Toolbox` as a demo of HSMC design

3.1.4 Homogeneous Proportional-Integral Control (HPIC)

Model of the control system:

$$\dot{x} = Ax + B(u + \gamma + p), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad A \in \mathbb{R}^{n\times m}, \quad B \in \mathbb{R}^{n\times m},$$

where the pair $\{A, B\}$ is controllable, $\gamma : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ is an unknown (vanishing at $x = \mathbf{0}$) function and $p \in \mathbb{R}^n$ is an unknown constant.

Control law:

$$u_{hpic} = u_{hpc} + \int_0^t u_{\text{int}}(x(\tau))d\tau$$

$$u_{hpc} = K_0 x + \|x\|_{\mathbf{d}}^{1+\mu} K\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x, \quad K_0 \in \mathbb{R}^{m\times n}, \quad K \in \mathbb{R}^{m\times n}$$

$$u_{\text{int}} = \|x\|_{\mathbf{d}}^{1+2\mu} \frac{K_i \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x}{x^\top \mathbf{d}^\top(-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x}, \quad K_i \in \mathbb{R}^{m\times n}$$

where $\mu \geq -0.5$, $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$ is a dilation in $\mathbb{R}^n$, $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ and the homogeneous norm $\|x\|_{\mathbf{d}}$ is induced by the weighted Euclidean norm $\|x\| = \sqrt{x^\top P x}$ in $\mathbb{R}^n$, $P \in \mathbb{R}^{n\times n}$.

Properties:

- finite-time stabilization for negative homogeneity degree $\mu < 0$
- nearly fixed-time stabilization for positive homogeneity degree $\mu > 0$
- rejection of the unknown constant perturbation p and matched (vanishing at $x = \mathbf{0}$) disturbance γ if $|\gamma(t, x)| \leq \gamma_{\max} \|x\|_{\mathbf{d}}^{1+\mu}$;
- generalized homogeneity of the augmented system for x and $x_{n+1} = p + \int u_{\text{int}}$.

HPIC Design

The function `hpic_design` computes parameters K_0, K, K_i, G_d and P of HPC for given A, B, $\mu \geq -0.5$, $\gamma_{\max} \geq 0$ and $\rho > 0$ (the parameter $\rho > 0$ can be utilized for tuning of convergence time: the larger ρ, the faster convergence).

- **Input parameters**: A, B, μ and ρ (by default $\rho = 1$) and $\gamma_{\max}$ (by default $\gamma_{\max} = 0$)
- **Output parameters**: K_0, K, K_i, G_d and P

HPIC Implementation

- The function `e_hpic` computes *explicit* discretization of u_{hpic}
  - **Input parameters**: x, K_0, K, μ, $G_{\mathbf{d}}$, P
  - **Output parameters**: u_{hpc} and u_{int}

Use `demo_hpc.m` from `HCS Toolbox` as a demo of HPC design

3.1.5 Fixed-time HPIC

Model of the control system:

$$\dot{x} = Ax + B(u+p), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad A \in \mathbb{R}^{n\times m}, \quad B \in \mathbb{R}^{n\times m},$$

where the pair $\{A, B\}$ is controllable and $p \in \mathbb{R}^n$ is an unknown constant.

Control law:

$$u_{hpic} = u_{fhpc} + \int_0^t u_{\text{fint}}(x(\tau))d\tau$$

$$u_{fhpc} = K_0 x + \begin{cases} \|x\|_{\mathbf{d}_1}^{1+\mu_1} K\mathbf{d}_1(-\ln\|x\|_{\mathbf{d}_1})x & \text{if} \quad x^\top P x \le 1 \\ \|x\|_{\mathbf{d}_2}^{1+\mu_2} K\mathbf{d}_2(-\ln\|x\|_{\mathbf{d}_2})x & \text{if} \quad x^\top P x > 1 \end{cases}$$

$$u_{\text{fint}} = \begin{cases} \frac{\|x\|_{\mathbf{d}_1}^{1+2\mu_1} K_i \mathbf{d}_1(-\ln\|x\|_{\mathbf{d}_1})x}{x^\top \mathbf{d}_1^\top(-\ln\|x\|_{\mathbf{d}_1}) P G_{\mathbf{d}_1} \mathbf{d}_1(-\ln\|x\|_{\mathbf{d}_1})x} & \text{if} \quad x^\top P x \le 1 \\ \frac{\|x\|_{\mathbf{d}_2}^{1+2\mu_2} K_i \mathbf{d}_2(-\ln\|x\|_{\mathbf{d}_2})x}{x^\top \mathbf{d}_2^\top(-\ln\|x\|_{\mathbf{d}_2}) P G_{\mathbf{d}_2} \mathbf{d}_2(-\ln\|x\|_{\mathbf{d}_2})x} & \text{if} \quad x^\top P x > 1 \end{cases}$$

where $\mu_1 < 0 < \mu_2$, $\mathbf{d}_k(s) = e^{s(I_n + \mu_k G_0)}$ and $\|x\|_{\mathbf{d}_k}$ is induced by $\|x\| = \sqrt{x^\top P x}$, $k = 1, 2$

Properties:

- fixed-time stabilization of linear plant:

$$\exists T_{\max} > 0 \quad : x(t) = \mathbf{0}, \quad \forall t \ge T_{\max}, \quad \forall x(0) \in \mathbb{R}^n$$

- rejection of the unknown constant perturbation p;
- local homogeneity of the augmented system for x and $x_{n+1} = p + \int u_{fint}$.

Fixed-time HPIC Design

The function `fhpic_design` computes parameters $K_0, K, K_i G_0, P, \mu_1, \mu_2$ of Fixed-time HPC for given system matrix A and control matrix B.

- **Input parameters**: A and B
- **Output parameters**: $K_0, K, K_i G_0, P, \mu_1, \mu_2$

Fixed-time HPIC Implementation

- The function `e_fhpic` computes *explicit* discretization of u_{fhpic}
  - **Input parameters**: x, K_0, K, K_i, μ_1, μ_2, G_0, P
  - **Output parameters**: u_{fhpic} and u_{fint}

Use `demo_fhpic.m` from `HCS Toolbox` as a demo of Fixed-time HPIC design

3.1.6 HSMC with integral action

Model of the control system:

$$\dot{x}=Ax+B(u+\gamma+p),\quad y=Cx,\quad x\in\mathbb{R}^n,\quad u\in\mathbb{R}^m,\quad A\in\mathbb{R}^{n\times m},\quad B\in\mathbb{R}^{n\times m},\quad C\in\mathbb{R}^{p\times n}$$

where $y\in\mathbb{R}^p$ is a controlled output, $\gamma:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^m$ is an unknown (vanishing at $x=\mathbf{0}$) function and $p\in\mathbb{R}^n$ is an unknown constant.

Control law:

$$u_{hsmci}=u_0+\int_0^t u_{\mathrm{sint}}d\tau$$

$$u_0=K_0x+\|Cx\|_{\mathbf{d}}^{0.5}K\mathbf{d}(-\ln\|Cx\|_{\mathbf{d}})Cx,\quad K_0\in\mathbb{R}^{m\times n},\quad K\in\mathbb{R}^{m\times p},$$

$$u_{\mathrm{sint}}(Cx(\tau))=K_i\mathbf{d}(-\ln\|Cx\|_{\mathbf{d}})Cx,$$

where $\mathbf{d}(s)=e^{sG_{\mathbf{d}}}$ is a dilation in $\mathbb{R}^p$, $G_{\mathbf{d}}\in\mathbb{R}^{p\times p}$ and the homogeneous norm $\|x\|_{\mathbf{d}}$ is induced by the weighted Euclidean norm $\|x\|=\sqrt{x^\top Px}$ in $\mathbb{R}^p$, $P\in\mathbb{R}^{p\times p}$.

Properties:

- enforces sliding mode on the surface $Cx=0$ in a finite time

$$\exists T=T(x(0)):Cx(t)=0,\quad \forall t\geq T$$

- rejection of the unknown constant perturbation p and the (vanishing at $x=\mathbf{0}$) matched perturbation γ if

$$|\gamma(t,x)|\leq\gamma_{\max}\|x\|_{\mathbf{d}}^{1/2}.$$

Design of HSMC with Integral action

The function `hsmc_design` computes parameters K_0,K,K_i,G_d and P of HPC for given A, B and $\gamma_{\max}\geq 0$

- **Input parameters**: A, B, ρ (by default $\rho=1$) and $\gamma_{\max}$ (by default $\gamma_{\max}=0$)
- **Output parameters**: K_0,K,G_d and P

Implementation of HSMC with integral action

- The function `e_hsmci` computes *explicit* discretization of u_{hsmci}
  - **Input parameters**: x, C, K_0,K,K_i, $G_{\mathbf{d}}$, P
  - **Output parameters**: u_0 and u_{sint}

Use `demo_hsmci.m` from `HCS Toolbox` as a demo of HSMC with Integral action

3.1.7 Upgrading Linear Proportional Controller (LPC) to HPC

Model of the control system:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad A \in \mathbb{R}^{n\times m}, \quad B \in \mathbb{R}^{n\times m}$$

$$u = K_{lin}x, \quad \text{(already well-tuned LPC)}$$

where the pair $\{A, B\}$ is controllable and $K_{lin} \in \mathbb{R}^{m\times n}$ is such that

the matrix $A + BK_{lin}$ is Hurwitz.

The aim is to upgrade LPC to HPC:

$$u_{hpc} = K_0 x + \|x\|_{\mathbf{d}}^{1+\mu}(K_{\text{lin}} - K_0)\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x \tag{3.2}$$

where $\mathbf{d}(s) = e^{s(I_n + \mu G_0)}$ - dilation, $\mu \in [\mu_{\text{min}}, \mu_{\text{max}}]$, $\|x\|_{\mathbf{d}}$ is induced by $\|x\| = \sqrt{x^\top P x}$

Upgrading LPC to HPC

The function `lpc2hpc` computes parameters K_0, G_0, P and $\mu_1 < 0 < \mu_2$ of HPC for given A, B and K_{lin}

- **Input parameters**: A, B and K_{lin}
- **Output parameters**: K_0, G_0, P, $\mu_{\text{min}} < 0 < \mu_{\text{max}}$

Implementation of HPC

See Section 3.1.1 for $G_{\mathbf{d}} = I_n + \mu G_0$ and $K = K_{lin} - K_0$.

Global Upgrading Algorithm

- Take $\mu = 0 \quad \Rightarrow \quad \tilde{u}_{hpc} = K_{\text{lin}}x$
- Decrease μ or increase μ while a control quality is improving

Local Upgrading Algorithm

- Use the saturation $\text{sat}_{a,b}(q) = \max(a, \min(b, q))$, where $0 \le a \le b \le +\infty$ to restrict the homogeneous norm

$$\tilde{u}_{hpc} = K_0 x + \text{sat}_{a,b} \|x\|_{\mathbf{d}}^{1+\mu}(K_{\text{lin}} - K_0)\mathbf{d}(-\ln \text{sat}_{a,b} \|x\|_{\mathbf{d}})x$$

- Take $a = b = 1 \quad \Rightarrow \quad \tilde{u}_{hpc} = K_{\text{lin}}x$
- Decrease a and increase b while a control quality is improving

Use `demo_lpc2hpc.m` from `HCS Toolbox` as a demo of LPC to HPC upgrade for Rotary Inverted Pendulum Quanser QUBE Servo-2

3.1.8 Upgrading Linear Proportional-Integral Controller (LPIC) to HPIC

Model of the control system:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad A \in \mathbb{R}^{n\times m}, \quad B \in \mathbb{R}^{n\times m}$$

$$u = K_{lin}x + \int_0^t K_i x(\tau)d\tau, \quad \text{(already well-tuned LPIC)}$$

where the pair $\{A, B\}$ is controllable and $K_{lin} \in \mathbb{R}^{m\times n}$,$K_i \in \mathbb{R}^{m\times n}$ are such that

$$\text{the matrix } \begin{pmatrix} A + BK_{lin} & B \\ K_{lin} & \mathbf{0} \end{pmatrix} \text{ is Hurwitz.}$$

The aim is to upgrade LPIC to HPIC:

$$u_{\text{hpic}} = u_{\text{hpc}}(x) + \int_0^t u_{\text{int}}(x(\tau))d\tau$$

$$u_{hpc} = K_0 x + \|x\|_{\mathbf{d}}^{1+\mu}(K_{\text{lin}} - K_0)\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x$$

$$u_{\text{int}} = \frac{\|x\|_{\mathbf{d}}^{1+2\mu} K_i^{new}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}{x^\top \mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})PG_{\mathbf{d}}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}$$

where $\mathbf{d}(s) = e^{s(I_n + \mu G_0)}$ is a dilation in $\mathbb{R}^n$ for any $\mu \in [\mu_1, \mu_2]$ and $\|x\|_{\mathbf{d}}$ is induced by $\|x\| = \sqrt{x^\top P x}$

Upgrading LPIC to HPIC

The function `lpic2hpic` computes parameters K_0, G_0, P, K_i^{new} and $\mu_1 < 0 < \mu_2$ of HPIC for given A, B and K_{lin}, K_i

- **Input parameters**: A, B and K_{lin}
- **Output parameters**: K_0, G_0, P, $\mu_1 < 0 < \mu_2$

Implementation of HPIC

See Section 3.1.4 for $G_{\mathbf{d}} = I_n + \mu G_0$ and $K = K_{lin} - K_0$.

Global/Local Upgrading Algorithm

See Section 3.1.7

Use `demo_lpic2hpic.m` from `HCS Toolbox` as a demo of LPIC to HPIC upgrade

3.2 Observers

3.2.1 Homogeneous Observer (HO)

Model of the system:

$$\dot{x} = Ax + p, \quad y = Cx \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad A \in \mathbb{R}^{n\times m}, \quad C \in \mathbb{R}^{k\times n}$$

where the pair $\{A, C\}$ is observable and $p \in \mathbb{R}^n$ - known exogenous input

Observer:

$$\dot{z} = Az + p + \left(L_0 + |Cz - y|^{\nu-1}\mathbf{d}(\ln|Cz - y|)L\right)(Cz - y), \quad L_0 \in \mathbb{R}^{n\times k}, \quad L \in \mathbb{R}^{n\times k}$$

where $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$ is a dilation in $\mathbb{R}^n$ and $\nu \geq -1/\tilde{n}$, where $\tilde{n} \in \mathbb{N}$ is a minimal natural number such that rank $\begin{bmatrix} C \\ CA \\ \ldots \\ CA^{n-1} \end{bmatrix} = n$.

Properties:

- finite-time state observation for $\nu < 0$:

$$\forall x(0) \in \mathbb{R}^n, \quad \exists T = T(x(0)) \quad : \quad z(t) = x(t), \quad \forall t \geq T$$

- nearly fixed-time state estimation for $\nu > 0$:

$$\forall r > 0, \quad \exists T_r > 0 \quad : \quad \|z(t) - x(t)\| \leq r, \quad \forall t \geq T_r, \quad \forall x(0) \in \mathbb{R}^n.$$

- the error $\epsilon = z - x$ has a **d**-homogeneous dynamics of degree ν $\Rightarrow$ ISS (Input-to-State Stability) with respect to measurement noises.

Design of HO

The function `ho_design` computes parameters L_0, L and $G_{\mathbf{d}}$ of HO

- **Input parameters**: A, C and $\nu > 0$
- **Output parameters**: L_0, L and $G_{\mathbf{d}}$

Implementation of HO

- The function `e_ho` computes *explicit* discretization of HO
  - **Input parameters**: h(sampling period),z, y, A, C, p, L_0, L, $G_{\mathbf{d}}$, ν
  - **Output parameters**: z^{new} - new estimation of x
- The function `si_ho` computes *semi-implicit* discretization of HO
  - **Input parameters**: h, z, y, A, C, p, L_0, L, $G_{\mathbf{d}}$, ν
  - **Output parameters**: z^{new} - new estimation of x

Use `demo_ho.m` from `HCS Toolbox` as a demo of HO design

3.2.2 Fixed-time HO

Model of the system:

$$\dot{x} = Ax + p, \quad y = Cx \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad A \in \mathbb{R}^{n\times m}, \quad C \in \mathbb{R}^{k\times n}$$

where the pair $\{A, C\}$ is observable and $p \in \mathbb{R}^n$ - known exogenous input

Observer:

$$\dot{z} = Az + p + \begin{cases} \left(L_0 + |Cz-y|^{\nu_1-1}\mathbf{d}_1(\ln|Cz-y|)L\right)(Cz-y) & \text{if} \quad |Cz-y| \le 1 \\ \left(L_0 + |Cz-y|^{\nu_2-1}\mathbf{d}_2(\ln|Cz-y|)L\right)(Cz-y) & \text{if} \quad |Cz-y| > 1 \end{cases}$$

where $\nu_1 < 0 < \nu_2$, $\mathbf{d}_k(s) = e^{s(I_n + \nu_k G_0)}$, $k=1,2$ and $L_0 \in \mathbb{R}^{n\times k}$, $L \in \mathbb{R}^{n\times k}$

Properties:

- fixed-time state observation:

$$\exists T_{\max} > 0 : \quad : \quad z(t) = x(t), \quad \forall t \ge T_{\max}, \quad \forall x(0) \in \mathbb{R}^n$$

- the error $\epsilon = z - x$ has a locally homogeneous dynamics $\quad\Rightarrow$
ISS (Input-to-State Stability) with respect to measurement noises.

Design of Fixed-time HO

The function `fho_design` computes parameters L_0, L and $G_{\mathbf{d}}$ of HO

- **Input parameters**: A, C
- **Output parameters**: L_0, L, G_0, ν_1, ν_2

Implementation of Fixed-time HO

- The function `e_fho` computes *explicit* discretization of Fixed-time HO
  - **Input parameters**: h,z, y, A, C, p, L_0, L, G_0, ν_1, ν_2
  - **Output parameters**: z^{new} - new estimation of x
- The function `si_fho` computes *semi-implicit* discretization of Fixed-time HO
  - **Input parameters**: h,z, y, A, C, p, L_0, L, G_0, ν_1, ν_2
  - **Output parameters**: z^{new} - new estimation of x

Use `demo_fho.m` from `HCS Toolbox` as a demo of HO design

3.2.3 Upgrading Linear Observer (LO) to HO

Model of the system:

$$\dot{x} = Ax + p, \quad y = Cx \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad A \in \mathbb{R}^{n\times m}, \quad C \in \mathbb{R}^{k\times n}$$

$$\dot{z} = Az + p + L_{lin}(Cz - y),$$

where the pair $\{A, C\}$ is observable and $p \in \mathbb{R}^n$ - known exogenous input and the gain of $L_{lin} \in \mathbb{R}^{n\times k}$ the linear observer is such that

the matrix $A + L_{lin}C$ is Hurwitz.

The **aim** is to upgrade LO to HO

$$\dot{z} = Az + p + \left(L_0 + |Cz - y|^{\nu-1}\mathbf{d}(\ln|Cz - y|)(L_{lin} - L_0)\right)(Cz - y), \quad L_0 \in \mathbb{R}^{n\times k},$$

where $\mathbf{d}(s) = e^{sI_n + \nu G_0}$ is a dilation in $\mathbb{R}^n$ and $\nu \in [\nu_{\min}, \nu_{\max}]$.

Upgrading LO to HO

The function `lo2ho` computes parameters L_0, $\nu_{\min}$, $\nu_{\max}$ and G_0 of HO

- **Input parameters**: A, C, L_{lin},
- **Output parameters**: L_0, $G_0, \nu_{\min}, \nu_{\max}$

Implementation of HO

See Section 3.2.1 for $G_{\mathbf{d}} = I_n + \nu G_0$ and $L = L_{lin} - L_0$.

Global Upgrading Algorithm

- Take $\nu = 0 \quad \Rightarrow \quad$ HO becomes LO
- Decrease μ or increase μ while a control quality is improving

Local Upgrading Algorithm

- Use the saturation $\mathrm{sat}_{a,b}(q) = \max(a, \min(b, q))$, where $0 \le a \le b \le +\infty$ to restrict the homogeneous norm

  $$\dot{z} = Az + p + \left(L_0 + \mathrm{sat}_{a,b}\,|Cz - y|^{\nu-1}\mathbf{d}(\ln \mathrm{sat}_{a,b}\,|Cz - y|)(L_{lin} - L_0)\right)(Cz - y),$$

- Take $a = b = 1 \quad \Rightarrow \quad$ HO becomes LO
- Decrease a and increase b while a control quality is improving

Use `demo_lo2ho.m` from `HCS Toolbox` as a demo of upgrading LO to HO for Rotary Inverted Pendulum Quanser QUBE Servo - 2

3.3 Miscellaneous functions

This section surveys supporting functions of HCS Toolbox induced by linear dilations.

3.3.1 Homogeneous Curves

Given $x \in \mathbb{R}^n$ the set

$$\Gamma_{\mathbf{d}}(x) = \{\mathbf{d}(s)x : s \in \mathbb{R}\}, \tag{3.3}$$

is called a **d**-homogeneous curve.

Design of a homogeneous curve

The function `hcurve` generates an array of points on the homogeneous curve crossing the point $x \in \mathbb{R}^n$.

- **Input parameters**: x, $G_{\mathbf{d}}$, s_l (array of points $s_i \in \mathbb{R}, i = 1, 2, ...$)
- **Output parameters**: x_l (array of points corresponding to s_l)

Use `demo_hcurve.m` from `HCS Toolbox` as a demo of plotting the homogeneous curve

3.3.2 Homogeneous Spheres

The set

$$S_{\mathbf{d}}(r) = \{z \in \mathbb{R}^n : \|z\|_{\mathbf{d}} = r\},$$

is a **d-homogeneous sphere** of the radius $r > 0$, where the dilation $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$ is a dilation monotone with respect to the norm $\|z\| = \sqrt{z^\top Pz}$ and $\|\cdot\|_{\mathbf{d}}$ is the canonical homogeneous norm induced by $\|\cdot\|$.

Design of a homogeneous sphere

The function `hsphere` generates an array of points on a homogeneous sphere of the radius r.

- **Input parameters**: r, $G_{\mathbf{d}}$, P and $N_{\max}$ (number of points to be randomly generated on the sphere)
- **Output parameters**: M (array of points on the sphere $S_{\mathbf{d}}(r)$)

Use `demo_hsphere.m` from `HCS Toolbox` as a demo of plotting the homogeneous spheres

3.3.3 Homogeneous Norm $\|\cdot\|_{\mathbf{d}}$

Computation of homogeneous norm

The function `hnorm` computes homogeneous norm of the vector x.

- **Input parameters**: x, $G_{\mathbf{d}}$, P
- **Output parameters**: $\|x\|_{\mathbf{d}}$

3.3.4 Homogeneous Projection

If $\mathbf{d}$ is a continuous dilation in $\mathbb{R}^n$ then for any $z \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ there exist $s_0 \in \mathbb{R}$ and $z_0 \in S$ such that $z_0 = \mathbf{d}(s_0)z$. The corresponding point $z_0 \in S$ is called a *homogeneous projection* of z on the unit sphere S.

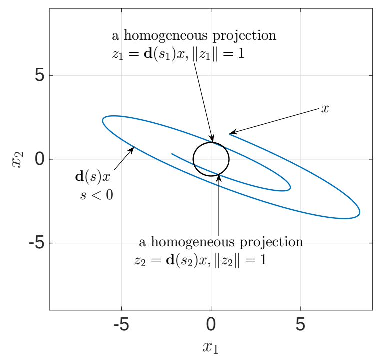


Figure 3.1: Illustration of homogeneous projection

If d is a monotone dilation then homogeneous projection is unique [15].

Computation of homogeneous projection

The function `hproj` computes homogeneous projection of the vector x to the unit sphere $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$ for a monotone dilation.

- **Input parameters**: x, $G_{\mathbf{d}}$, P
- **Output parameters**: $z \in S$ (homogeneous projection of x)

Chapter 4

Use Case

Int his chapter a procedure of upgrading linear controller to homogeneous one for an existing/operating system (Rotary Inverted Pendulum Quanser QUBE Servo - 2) is demonstrated.

4.1 Model of the system

A schematic representation of the rotary inverted pendulum (IP) is shown in Figure 4.1. The generalized coordinates θ and α describe the angular positions of the rotary arm and the pendulum, respectively. To obtain motion equations, the pendulum is considered as a lumped mass at its center.

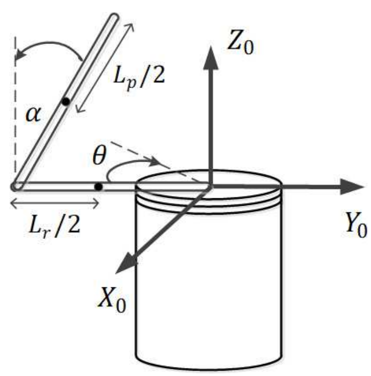


Figure 4.1: Schematic diagram of inverted pendulum (IP)

Table 4.1 presents the notation utilized for model description.

The dynamic model of the inverted pendulum is derived by means of the Euler-Lagrange method:

$$\frac{\partial^2 L}{\partial t \partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i. \tag{4.1}$$

Symbol	Description
m_p	Mass of the pendulum
L_p	Length of the pendulum
J_p	Inertia of the pendulum
D_p	Pendulum damping coefficient
L_r	Length of the rotary arm
J_r	Rotary arm inertia
D_r	Viscous damping coefficient
g	Gravitational acceleration

Table 4.1: Parameters of the rotary inverted pendulum

The Lagrangian of the pendulum is described as:

$$L = T - V \tag{4.2}$$

where T is the total kinetic energy of the inverted pendulum and V is the total potential energy of the system.

The variable q_i represents the generalized coordinates, in our case, given by

$$q(t) := \begin{bmatrix} \theta(t) & \alpha(t) \end{bmatrix}^\top \tag{4.3}$$

Considering the defined generalized coordinates (4.3), the Euler-Lagrange equations become:

$$\frac{\partial^2 L}{\partial t \partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q_1 \qquad \frac{\partial^2 L}{\partial t \partial \dot{\alpha}} - \frac{\partial L}{\partial \alpha} = Q_2 \tag{4.4}$$

The generalized forces Q_i describe non-conservative forces applied to the system. In our case, the generalized forces acting on the IP are:

$$Q_1 = \tau - D_r\dot{\theta} \qquad Q_2 = -D_p\dot{\alpha} \tag{4.5}$$

Once the kinetic and potential energy are obtained, then the Lagrangian is found, the nonlinear dynamic equations of motion for the inverted pendulum are:

$$\begin{gathered} \left(\psi + 0.25\zeta - 0.25\zeta\cos(\alpha)^2 + J_r\right)\ddot{\theta} - 0.5\varpi\cos(\alpha)\ddot{\alpha} \\ +0.5\zeta\sin(\alpha)\cos(\alpha)\dot{\theta}\dot{\alpha} + 0.5\varpi\sin(\alpha)\dot{\alpha}^2 = \tau - D_r\dot{\theta} \end{gathered} \tag{4.6}$$

and

$$\begin{gathered} -0.5\varpi\cos(\alpha)\ddot{\theta} + (J_p + 0.25\zeta)\,\ddot{\alpha} - 0.25\zeta\cos(\alpha)\sin(\alpha)\dot{\theta}^2 \\ -0.5m_pL_pg\sin(\alpha) = -D_p\dot{\alpha} \end{gathered} \tag{4.7}$$

where $\zeta = m_pL_p^2$, $\psi = m_pL_r^2$ and $\varpi = m_pL_pL_r$.

A torque generated by the servo motor and applied to the rotary arm is described by the following equation:

$$\tau = \tfrac{k_m(V_m - k_m\dot{\theta})}{R_m}, \tag{4.8}$$

where k_m is the motor back EMF (electromotive force) constant, R_m is the terminal resistance and V_m is the control input (the input voltage for the servo motor).

Notice that for a generalized coordinate vector $q(t)$, the equations (4.6) and (4.7) can be transformed in the following matrix form:

$$J(q)\ddot{q} + C(q,\dot{q})\dot{q} + w(q) = \phi \tag{4.9}$$

where

$$J(q) = \begin{bmatrix} Jr + \psi + \frac{1}{4}(\zeta - \zeta\cos^2(\alpha)) & -\frac{1}{2}\varpi\cos(\alpha) \\ -\frac{1}{2}\varpi\cos(\alpha) & J_p + \frac{1}{4}\zeta \end{bmatrix}$$

$$w(q) = \begin{bmatrix} 0 \\ -\frac{1}{2}gm_pL_p\sin(\alpha) \end{bmatrix}, \quad \phi = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \tag{4.10}$$

$$C(q) = \begin{bmatrix} \frac{1}{2}\zeta\sin(\alpha)\cos(\alpha)\dot{\alpha} + D_r & \frac{1}{2}\varpi\sin(\alpha)\dot{\alpha} \\ -\frac{1}{4}\zeta\sin(\alpha)\cos(\alpha)\dot{\theta} & D_p \end{bmatrix}$$

The **control aim** is to stabilize the pendulum arm at the upper position ($\alpha = 0$)

The conventional approach [3] to solving this control problem is switching between two control strategies:

- swing-up control accumulates an kinetic energy (increases oscillation amplitude) of the pendulum to bring it close to the upper position
- a stabilizing feedback operating locally (close to the upper position)

The swing-up control is usually defined as an optimal feed-forward (or feedback) algorithm [3], so a **stabilizing controller is needed to be designed** only.

To design the stabilizing feedback the nonlinear model of the inverted pendulum is usually linearized around the operating point $\alpha = 0$ (upper position of the pendulum) using the equations (4.6) and (4.7).

Let $x_1 = \theta$, $x_2 = \alpha$, $x_3 = \dot{\theta}$ and $x_4 = \dot{\alpha}$. From the equation (4.9) for α close to zero we have $\sin(x_2) \approx 0$, $\cos(x_2) \approx 1$, then the linearized state-space representation for the inverted pendulum satisfies the following differential equations:

$$\begin{gathered} \dot{x}_1 = x_3, \qquad \dot{x}_2 = x_4 \\ \dot{x}_3 = \frac{-\left(J_p + 0.25\zeta\right)D_r x_3 - 0.5\varpi D_p x_4 + 0.25\vartheta g x_2 + \left(J_p + 0.25\zeta\right)\tau}{J_T} \\ \dot{x}_4 = \frac{0.5\varpi D_r x_3 - \left(J_r + \psi\right)D_p x_4 + 0.5m_pL_pg\left(J_r + \psi\right)x_2 + 0.5\varpi\tau}{J_T} \end{gathered} \tag{4.11}$$

where $J_T = J_p\zeta + J_rJ_p + 0.25J_r\zeta$ and $\vartheta = m_p^2L_p^2L_r$ and

$$\tau = \tfrac{k_m(u - k_m x_3)}{R_m}, \tag{4.12}$$

with $u = V_m$ being a control input (voltage).

Therefore, the linearized model of the inverted pendulum admits the following state-space representation:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4.13}$$

where $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^\top \in \mathbb{R}^4$ is the state vector, $u(t) \in \mathbb{R}$ is the control signal, $A \in \mathbb{R}^{4\times 4}$, $B \in \mathbb{R}^{4\times 1}$. In the equation (4.13), the matrices A, B are defined as:

$$A = \frac{1}{J_T}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_{3,2} & a_{3,3} & a_{3,4} \\ 0 & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}, \qquad B = \frac{k_m}{J_T R_m}\left(0,\ 0,\ (J_p + 0.25\zeta),\ 0.5\varpi \right)^\top \tag{4.14}$$

The elements of the matrix A are given by:

$$\begin{gathered} a_{3,2} = 0.25\vartheta g, \quad a_{3,3} = (J_p + 0.25\zeta)\,(k_m^2/R_m - D_r) \\ a_{3,4} = -0.25\varpi D_p, \quad a_{4,2} = 0.25 m_p L_p g (J_r + \psi) \\ a_{4,3} = 0.25\varpi (D_r - k_m^2/R_m), \quad a_{4,4} = -\,(J_r + \psi)\,D_p \end{gathered} \tag{4.15}$$

The pair $\{A, B\}$ is controllable in any realistic scenario.

4.2 Description of Experimental Setup

The platform QUBE- Servo 2 of Quanse (see Fig. 4.2) is utilized for the control upgrading experiment.

Figure 4.2: Rotary IP Quanser QUBE - Servo 2

The parameters of the experimental platform are given by the manufacturer and listed in the Table 4.2. The control input (voltage) is saturated by $\pm 10V$, i.e.

Parameter	Value
m_p	0.024 Kg
L_p	0.129 m
J_p	3.3x10^{-5} Kg m^2
D_p	0.0015 N m s/rad
L_r	0.085 m
J_r	5.7x10^{-5} Kg m^2
D_r	0.0005 N m s/rad
g	9.81 m/s^2

Table 4.2: Parameters of Quanser QUBE-Servo 2

$$u \in [-10, 10].$$

The Quanser's platform is supported with both a swing-up controller and a linear stabilizing controller realized in MATLAB. Our aim is to upgrade the linear stabilizing controller. The gains of the linear feedback (given by the manufacturer) are as follows

$$K_{lin} = \begin{pmatrix} 2 & -35 & 1.5 & -3 \end{pmatrix} \tag{4.16}$$

4.3 Upgrading linear controller via HCS Toolbox

The demo[1] of an upgrading linear Quanser's controller to homogeneous one is given in demo_lpc2hpc.m of HCS Toolbox.

1) The first part of the code defines the parameters of the rotary inverted pendulum (according to Table 4.2 provided by the manufacturer).

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Pendulum model
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Motor

% Resistance
Rm = 8.4;

% Current-torque (N-m/A)
kt = 0.042;

% Back-emf constant (V-s/rad)
km = 0.042;

%% Rotary Arm
```

[1] To open a demo, please type 'edit <name_of_demo>.m' in the Command Line of MATLAB

```
% Mass (kg)
mr = 0.095;

% Total length (m)
r = 0.085;

% Moment of inertia about pivot (kg-m^2)
Jr = mr*r^2/3;

% Equivalent Viscous Damping Coefficient (N-m-s/rad)
br = 1e-3; % damping tuned heuristically to match QUBE-Sero 2 response

%% Pendulum Link

% Mass (kg)
mp = 0.024;

% Total length (m)
Lp = 0.129;

% Pendulum center of mass (m)
l = Lp/2;

% Moment of inertia about pivot (kg-m^2)
Jp = mp*Lp^2/3;

% Equivalent Viscous Damping Coefficient (N-m-s/rad)
bp = 5e-5; % damping tuned heuristically to match QUBE-Sero 2 response

% Gravity Constant
g = 9.81;

% Total Inertia
Jt = Jr*Jp - mp^2*r^2*l^2;
```

2) Next, the parameters linearized model of the system (4.13) are computed.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Linearized model of the rotary inverted pendulum in the upper position
%%
%%          dx/dt=Ax+Bu,      x=(x1,x2,x3,x4)'
%%
%% where     x1  - angle of the pentulum arm
%%           x2  - angle of the rotary arm
%%           x3  - angular velocity of the pendulum arm
%%           x4  - angular velocity of the rotary arm
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
A = [0 0 1 0;
0 0 0 1;
0 mp^2*l^2*r*g/Jt  -br*Jp/Jt   -mp*l*r*bp/Jt
0  mp*g*l*Jr/Jt    -mp*l*r*br/Jt   -Jr*bp/Jt];
%
B = [0; 0; Jp/Jt; mp*l*r/Jt];

% adding a model of actuator dynamics
A(3,3) = A(3,3) - km*km/Rm*B(3);
A(4,3) = A(4,3) - km*km/Rm*B(4);
B = km * B / Rm;

 % the linear feedback gain (provided by manufacturer)
Klin=[2 -35 1.5 -3];
```

The system matrix A is

```
>> A

A =

0         0    1.0000         0
0         0         0    1.0000
0  152.0057  -12.2542   -0.5005
0  264.3080  -12.1117   -0.8702
```

The control matrix B is

```
>> B

B =

0
0
50.6372
50.0484
```

3) Finally, the parameters of the homogeneous controller are obtained using the function lpc2hpc of HCS Toolbox.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% HPC/FHPC design by upgrading a linear controller
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

[K0 G0 P mu_min mu_max]=lpc2hpc(A,B,Klin); % upgrade linear control to HPC

%selection of the homogeneity degree mu_min<= mu <=mu_max

Gd=eye(4)+mu_min*G0;  mu=mu_min; % for HCP with negative homogeneity degree
%Gd=eye(4)+mu_max*G0; mu=mu_max; % for HCP with positive homogeneity degree
% for FHCP use G0 mu_min mu_max
```

```
K=Klin-K0;

%K0 - homogenization feedback gain
%K  - control gain
%Gd - generator of dilation
%P  - shape matrix of the weighted Euclidean norm
```

The obtained parameters of homogeneous controller are

```
>> K0

K0 =

-0.0000   -5.2811    0.2420    0.0174

>> G0

G0 =

-3.0000    2.0248   -0.0033    0.0033
0.0000   -1.0000    0.0000   -0.0000
-0.0000    0.3800   -2.0000    2.0235
-0.0000   -0.0000    0.0000   -0.0000

>> P

P =

3.1581   -7.2001    0.5908   -0.6077
-7.2001   96.7019   -6.8570    7.4704
0.5908   -6.8570    0.5539   -0.5856
-0.6077    7.4704   -0.5856    0.6472

>> mu_min

mu_min =

-1

>> mu_max

mu_max =

0.1607
```

4) The rest of the code of demo_lpc2hpc is devoted to comparison of linear and homogeneous controllers on simulations

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Numerical Simulation
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

t=0; Tmax=3;
h=0.001; % sampling period

x=[1;1;0;0];
tl=[t];xl=[x];ul=[];

%alpha and beta are tuning parameters (alpha=beta=1 => linear control)

alpha=0.1; beta=1;  %for HPC with negative degree (upgrade close to zero)
%alpha=1;beta=100;   %for HPC with positive degree (upgrade close to Inf)
%alpha=0.1;beta=100; %for FHPC (global upgrade)

noise=0; %magnitude of measurement noises (may be changed for comparison)

disp('Run numerical simulation...');

[Ah Bh]=ZOH(h,A,B); %discretization of linear plant by ZOH

while t<Tmax

xm=x+2*noise*(rand(4,1)-0.5);       %modeling of noised measurement

%u=Klin*xm;                            %linear control (for comparison)

u=e_hpc(xm,K0,K,Gd,P,mu,alpha,beta);             %explicit HPC

%simulation of the system (with control saturation as in QUBE Servo-2)

x=Ah*x+Bh*min(10,max(-10,u));

t=t+h; tl=[tl t]; xl=[xl x]; ul=[ul u];

end;

ul=[ul u];

disp('Done!');

%%norm of the state at the time instant Tmax
disp(['||x(Tmax)||=',num2str(norm(x))])
```

The comparison of the simulation results for linear control (LC)

```
>> demo_lpc2hpc
Run numerical simulation...
Done!
||x(Tmax)||=0.0057923
```

with locally homogeneous controller (HPC)

```
>> demo_lpc2hpc
Run numerical simulation...
Done!
||x(Tmax)||=4.6229e-08
```

shows an essential improvement (in times, see above) of stabilization precision $|x(T)|$ at the terminal instant of time *without any degradation of the system transient* (see Fig.4.3 and Fig.4.4). The upgrade was done locally (close to $\mathbf{0}$). That is why trajectories of system with linear and homogeneous controller simply coincide on the time interval $[0, 1]$. The zoomed plots for the time interval $[1, 3]$ are depicted in Fig.4.5 and Fig.4.6, which clearly show faster convergence of the system with homogeneous controller.

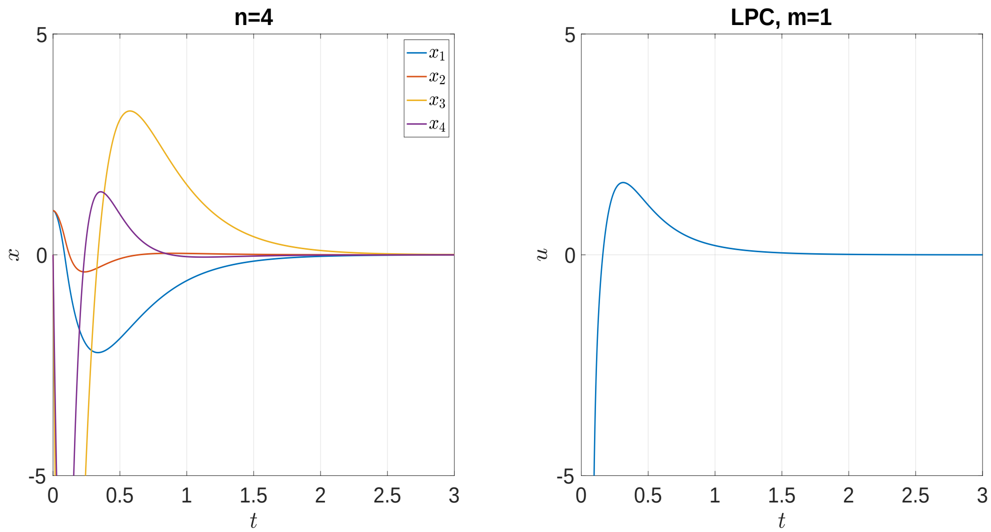


Figure 4.3: Simulation of linear proportional control (LPC) $u = K_{lin}x$ for linearized system (4.13)

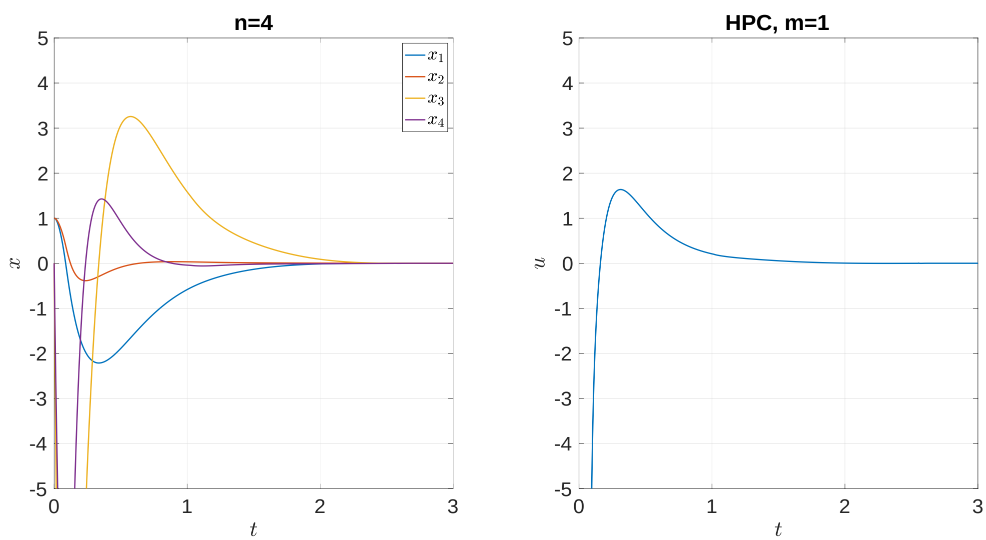


Figure 4.4: Simulation of locally homogeneous proportional controller (HPC) (3.2) for linearized system (4.13)

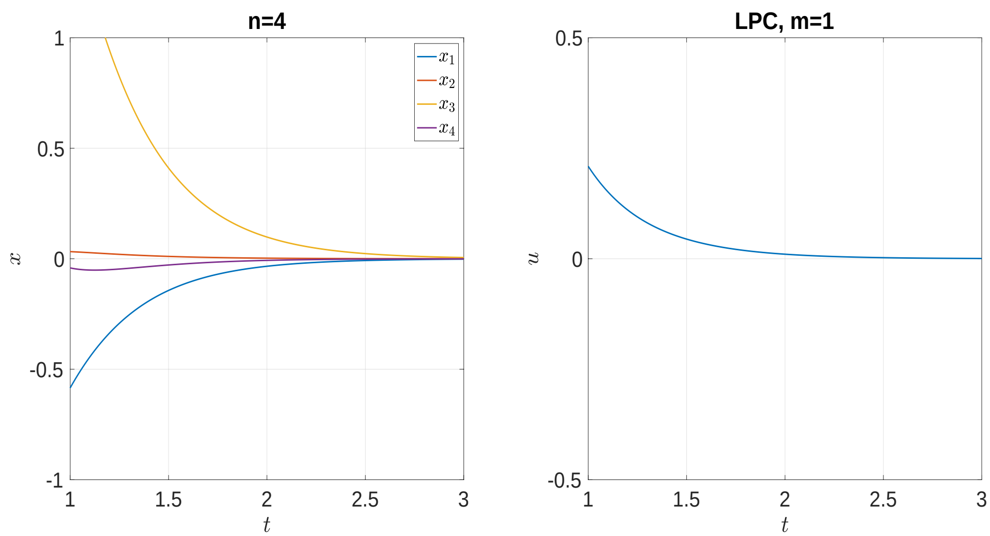


Figure 4.5: Simulation of linear proportional control (LPC) $u = K_{lin}x$ for linearized system (4.13) (zoom for the time interval $[1, 3]$)

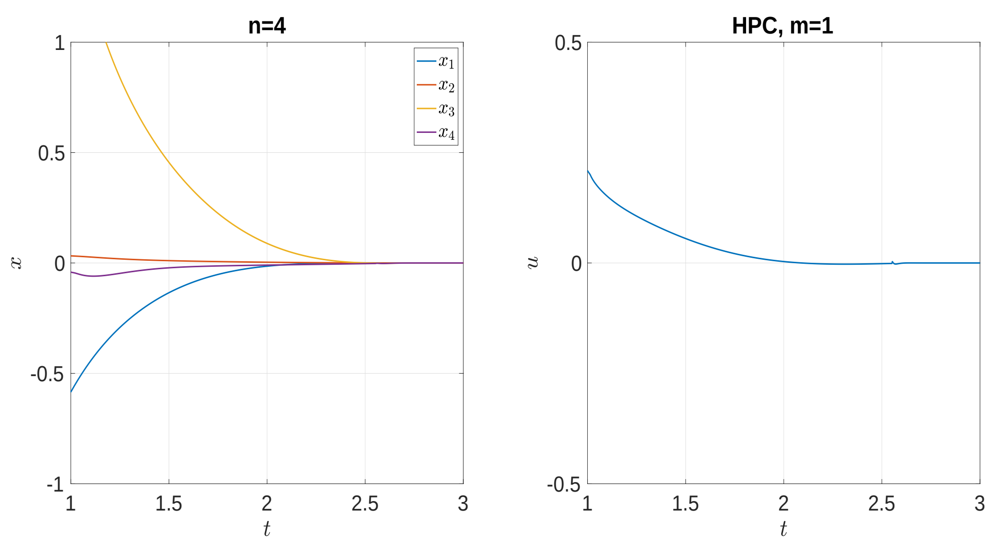


Figure 4.6: Simulation of locally homogeneous proportional controller (HPC) (3.2) for linearized system (4.13) (zoom for the time interval $[1, 3]$)

4.4 Comparison on real experiment in ControlHub

The setup for real control experiments with rotary inverted pendulum is available on-line (see Fig. 4.7) and http://valse-pendulum.lille.inria.fr:5000 as a part of the ControlHub platform (under construction in Inria Lille).

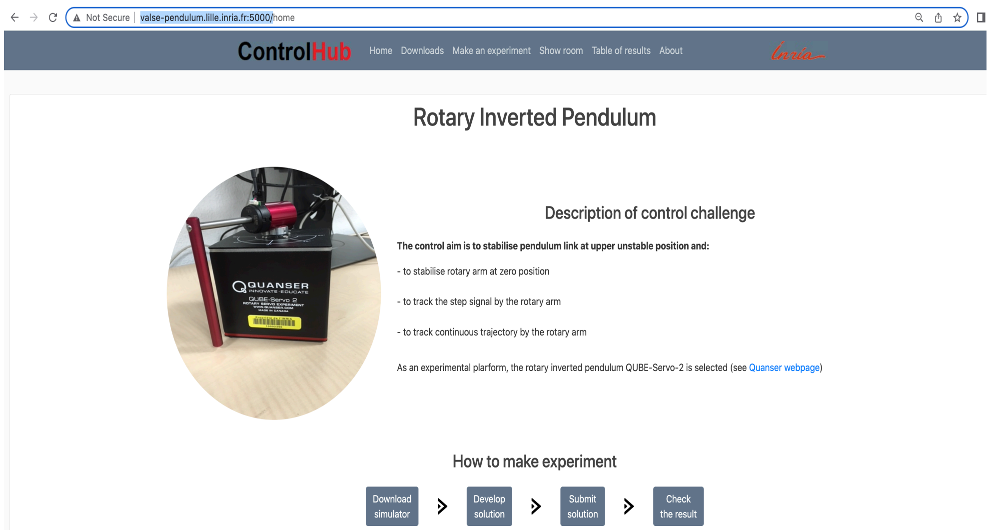


Figure 4.7: Home page of rotary inverted pendulum control experiment

The platform is aimed at remote rapid prototyping, demonstration and comparison of various control algorithms on real experimental setups.

The user can upload its own control algorithm to the platform in order to test it on a real setup (Rotary Inverted Pendulum Quanser QUBE Servo - 2). The description of the experimental setup (provided by the manufacturer) and a Simulink model of the system for off-line validation of user-defined controllers can be found in the tab "Download".

To make an experiment, first, the user need to select a control task on which the user-defined control algorithm should be tested. There tasks are presently available:

- **Task 1**: stabilization ($\alpha \to 0, \theta \to 0$)
- **Task 2**: set-point tracking ($\alpha \to 0$, $\theta \to$ step signal):

$$\theta_{ref} = \begin{cases} 0 & \text{if } t \leq 5 \\ \pi/5 & \text{if } 5 < t \leq 10 \\ -\pi/5 & \text{if } 10 < t \leq 15 \\ 0 & \text{if } t > 15 \end{cases}$$

- **Task 3**: continuous trajectory tracking ($\alpha \to 0, \theta \to$ sinusoidal signal)

$$\theta_{ref}(t) = 0.3 \sin(t)$$

Next, the control/observer algorithm should be uploaded or directly typed (pasted) in the forms on the bottom of the page (see Fig. 4.8).

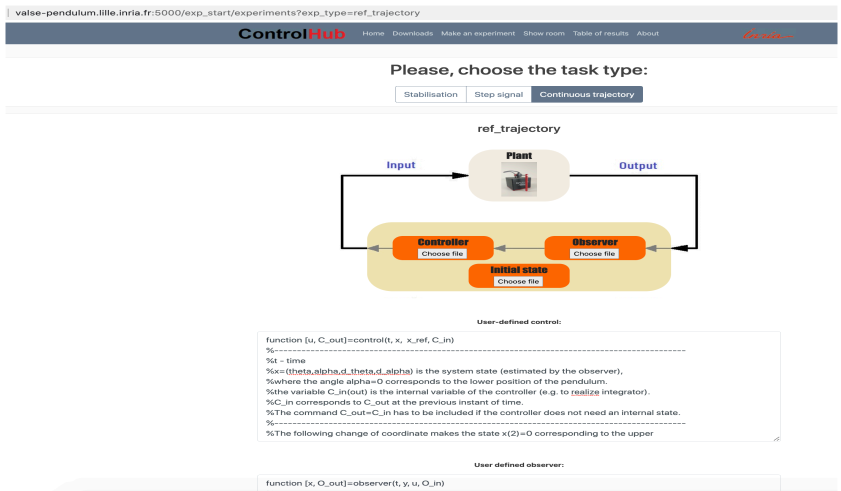


Figure 4.8: Submission of the control law for a testing

The control algorithm has to be realized as m-function of MATLAB in a certain format. By default, the linear (Quanser's) controller is implemented:

```
function [u, C_out]=control(t, x,  x_ref, C_in)
%-------------------------------------------------------------------------------
%t - time
%x=(theta,alpha,d_theta,d_alpha) is the system state (estimated by the observer),
%where the angle alpha=0 corresponds to the lower position of the pendulum.
%the variable C_in(out) is the internal variable of the controller
%C_in corresponds to C_out at the previous instant of time.
%The command C_out=C_in has to be included if the controller does not need an
% internal state.
%------------------------------------------------------------------------------
%The following change of coordinate makes the state x(2)=0 corresponding to
%the upper position of the pendulum.
 x(2)=mod (x(2),2*pi)-pi;
 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% THE USER-DEFINED CONTROLLER HAS TO BE REALIZED BELOW
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 C_out=C_in;
 %The static linear feedback u=K(x-x^ref) stabilizes the pendulum in
%the upper position and tracks the reference theta^ref(t)=0.3sin(t).

 u=[2 -35 1.5 -3]*(x-x_ref);
```

Click on the button 'Submit' on the bottom of the page to send your control solution for testing. To identify your own control solution in the list of other submission, it is recommended put some name to the field 'Insert your submission name'.

The user's controller/observer will be first tested on simulator and next on the real experimental setup (Quanser QUBE Servo-2). The whole process takes about 3 minutes. In the tab "Show room" the user can survey the testing progress on-line using the web-camera installed in the lab. The results will be added to the database (the tab 'Table of results'). The user can download the results and compare the results of the submitted controller with the results for other controllers stored in the database.

Since the HCS Toolbox is already installed on the pendulum platform, then its functions can be utilized for implementation of HPC[2]:

```
function [u, C_out]=control(t, x,  x_ref, C_in)
%-------------------------------------------------------------------------------
%t - time
%x=(theta,alpha,d_theta,d_alpha) is the system state (estimated by the observer),
%where the angle alpha=0 corresponds to the lower position of the pendulum.
%the variable C_in(out) is the internal variable of the controller
%C_in corresponds to C_out at the previous instant of time.
%The command C_out=C_in has to be included if the controller does not need an
% internal state.
%------------------------------------------------------------------------------
%The following change of coordinate makes the state x(2)=0 corresponding to
%the upper position of the pendulum.

x(2)=mod (x(2),2*pi)-pi;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% THE USER-DEFINED CONTROLLER HAS TO BE REALIZED BELOW
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

C_out=C_in;

%The static linear feedback u=K(x-x_ref) stabilizes the pendulum in
%the upper position and tracks the reference theta^ref(t)=0.3sin(t).

K0=[-0.0000   -5.2811    0.2420    0.0174];

Gd=[  4.0000   -2.0248    0.0033   -0.0033;
-0.0000    2.0000   -0.0000    0.0000;
0.0000   -0.3800    3.0000   -2.0235;
0.0000    0.0000   -0.0000    1.0000];

K=[2.0000  -29.7189    1.2580   -3.0174];

P=[
3.1581   -7.2001    0.5908   -0.6077;
-7.2001   96.7019   -6.8570    7.4704;
0.5908   -6.8570    0.5539   -0.5856;
-0.6077    7.4704   -0.5856    0.6472];

u=e_hpc(x-x_ref,K0,K,Gd,P,-1,0.8,1);
```

[2]The HPC discretization algorithms from HCS Toolbox are not optimized for implementation in low performance control devices. Concerning industrial implementation of homogeneous algorithms please contact the HCS Toolbox developer andrey.polyakov@inria.fr

The LPC and HPC controllers are locally stabilizing controllers (for the rotary pendulum), so they start to operate that the swing-up controller bring the system close to the upper position of the pendulum. That is why the comparison is made only on the time interval $[3, 20]$. If a stabilizing controller cannot hold the pendulum close to the upper position for $t \geq 3$ then the quality of the stabilizing controller is unacceptable. Both LPC and (upgraded) HPC solves successfully all three control tasks without any a-priori knowledge about reference trajectories. Both controllers are saturated by ± 10 V (due to physical restrictions of the motor's input voltage). The control input and output measurements are sampled with the period 0.002. Notice that only the angles α and θ are directly measured by encoder having certain quantization, but their derivatives are obtained by an observer/filter. For both controller the same state observer (provided by Quanser) is utilized.

The comparison results for LPC and HPC control application in the real device (Qunser QUBE Servro-2) are shown in Fig. 4.9 (Task 1), in Fig . 4.10 (Task 2) and in Fig . 4.11 (Task 3). In all three cases the tracking error of HPC is twice smaller than the tracking error of LPC. HPC also demonstrates smaller overshoots Fig . 4.10). However, the homogeneous controller consumes a bit more energy. Both L_2 and L_∞ norms of HPC signal are larger than in the case of LPC. This is also expectable, since faster transient needs additional power. There always exists a trade of between fast response+high precision and a consumption of control energy. Homogeneity provides a possible way to adjust this in practice (using the saturation parameters a, b, see Section 3.1.7).

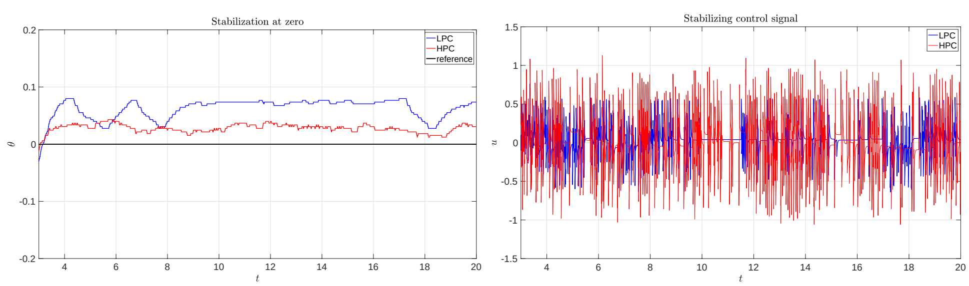


Figure 4.9: Comparison of LPC and HPC for Task 1

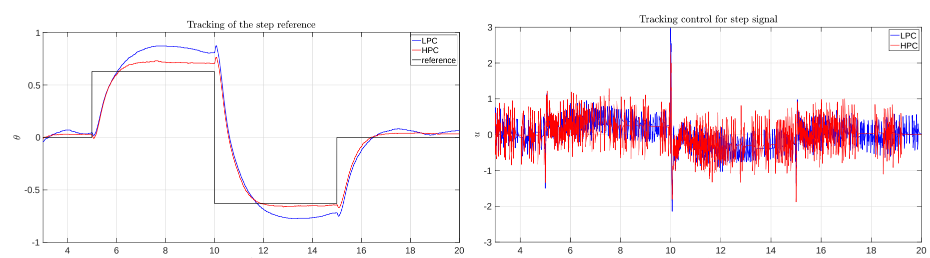


Figure 4.10: Comparison of LPC and HPC for Task 2

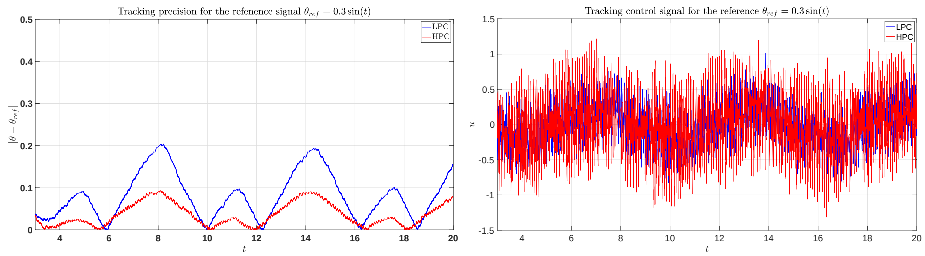


Figure 4.11: Comparison of LPC and HPC for Task 3

4.5 Conclusions

- Theoretical conclusions of Chapter 2 (about faster convergence, better robustness and smaller overshoots) for homogeneous systems are confirmed by practical control experiments.
- HCS Toolbox provides simple-in-use functions for design of homogeneous controllers/observer and upgrading existing linear algorithms to homogeneous ones.

Chapter 5

List of Acronyms

- HCS (Homogeneous Control System)
- HO (Homogeneous Observer)
- HPC (Homogeneous Proportional Control)
- HPIC (Homogeneous Proportional-Integral Control)
- HSMC (Homogeneous Sliding Mode Control)
- HSMCI (Homogeneous Sliding Mode Control with Integral action)
- LO (Linear Observer)
- LPC (Linear Proportional Control)
- LPIC (Linear Proportional-Integral Control)
- MIMO (Multiply-Input Multiply-Output)
- SISO (Single Inout Single Output)
- ZOH (Zero-Order-Hold)

Chapter 6

List of Functions of HCS Toolbox

(for more info type `help <name_of_function>` in the Command Line of MATLAB)

Homogeneous Objects induced by Linear Dilation:

- `hnorm` - computation of homogeneous norm
- `hproj` - computation of homogeneous projection
- `hcurve` - generation of points of a homogeneous curve
- `hsphere` - generation of a random grid on a homogeneous sphere

Homogeneous Control Design:

- `hpc_design` - Homogeneous Proportional Control (HPC) design
- `hpci_design` - Homogeneous Proportional-Integral Control (HPIC) design
- `hsmc_design` - Homogeneous Sliding Mode Controller (HSMC) design
- `hsmci_design` - design of HSMC with Integral action
- `fhpc_design` - Fixed-time HPC design
- `fhpic_design` - Fixed-time HPIC design
- `lpc2hpc` - upgrading Linear Proportional Control (LPC) to HPC
- `lpic2hpc` - upgrading Linear PI control (LPIC) to HPIC

Discretization of Homogeneous Control:

- `e_hpc` - explicit discretization of HPC
- `si_hpc` - semi-implicit discretization of HPC

- `c_hpc` - consistent discretization of HPC
- `e_hpic` - explicit discretization of HPIC
- `e_hsmc` - explicit discretization of HSMC
- `si_hsmc` - semi-implicit explicit discretization of HSMC
- `e_hsmci` - explicit discretization of HSMC with Integral action
- `e_fhpc` - explicit discretization of Fixed-time HPC
- `si_fhpc` - semi-implicit discretization of Fixed-time HPC
- `e_fhpic` - explicit discretization of Fixed-time HPIC

Homogeneous Observer Design:

- `ho_design` - Homogeneous Observer (HO) design
- `fho_design` - Fixed-time HO design
- `lo2ho` - upgrading Linear Observer (LO) to HO

Discretization of Homogeneous Observer:

- `e_ho` - explicit Euler discretization of HO
- `si_ho` - semi-implicit discretization of HO
- `e_fho` - explicit Euler discretization of FHO
- `si_fho` - semi-implicit discretization of HO

Block forms:

- `block_con` - transformation to block controlability form
- `bloc_obs` - transformation to block observability form
- `trans_con` - transformation to partial block controlability form
- `trans_con` - transformation to partial block observability form
- `output_form` - transformation to reduced order output control system

Examples (to open type `edit <name_of_example>` in Command Line of MATLAB):

- `demo_hnorm` - demo of computation of a homogeneous norm
- `demo_hsphere` - plot of homogeneous spheres in 2D

- `demo_hcurve` - plot of homogeneous curves in 2D
- `demo_hpc` - demo of HPC design and simulation
- `demo_hpic` - demo of HPIC design and simulation
- `demo_hsmc` - demo of HSMC design and simulation
- `demo_hsmci` - demo of HSMCI design and simulation
- `demo_fhpc` - demo of FHPC design and simulation
- `demo_fhpic` - demo of FHPIC design and simulation
- `demo_lpc2hpc` - demo of upgrading LPC to HPC/FHPC
- `demo_lpic2hpic` - demo of upgrading LPIC to HPIC/FHPIC
- `demo_ho` - demo of HO design and simulation
- `demo_fho` - demo of FHO design and simulation
- `demo_lo2ho` - demo of upgrading LO to HO/FHO

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